ARITHMETIC DEFORMATIONS OF CRYSTALLINE COHOMOLOGY

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Abstract

We survey a new approach to constructing a deformation of crystalline cohomology for arithmetic schemes, motivated by archimedean and p-adic Hodge theory.

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CHAPTER 1

Introduction

In this first chapter we illustrate some of the reasons why this work was first conceived.

For a smooth proper scheme X over a number field K, there is a wealth of cohomology theories attached to X, as well as comparisons among them. For example, if we fix an embedding $\sigma : K \to \mathbf{C}$ then we can define $X(\mathbf{C}) = (X \otimes_{K,\sigma} \mathbf{C})(\mathbf{C})$ and consider its singular cohomology $H^*(X(\mathbf{C}), \mathbf{Q})$ or scalar extensions $H^*(X(\mathbf{C}), \mathbf{Q}_p)$ for primes p. The former is of essentially topological nature, while the latter has a purely algebro-geometric description thanks to the Artin comparison theorem:

$$H^*(X(\mathbf{C}), \mathbf{Q}_p) \simeq H^*_{\text{ét}}(X_{\bar{K}}, \mathbf{Q}_p).$$

Note that the target has a natural action by $\mathrm{Gal}(\bar{K}/K)$ that is difficult to perceive on the left side (due to its reliance on σ).

There is also the algebraic de Rham cohomology $H^*_{\text{dR}}(X/K)$ that consists of filtered K-vector spaces. Although its dimension over K agrees with that of $H^*(X(\mathbf{C}), K)$, there is no natural isomorphism between these. However, if we extend scalars to $\mathbf{C}$ then analytic tools provide a link via the de Rham comparison theorem:

$$H^*_{\text{dR}}(X/K) \otimes_K \mathbf{C} \simeq H^*_{\text{dR}}(X_{\mathbf{C}}/\mathbf{C}) \simeq H^*(X(\mathbf{C}), \mathbf{C}) = H^*(X(\mathbf{C}), \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}.$$

We will revisite this from a broader viewpoint shortly, in terms of vector bundles over a disc.

One of the aims of p-adic Hodge theory is to provide an analogous isomorphism relating $H^*_{\text{dR}}(X/K)$ and $H^*_{\text{ét}}(X_{\bar{K}}, \mathbf{Q}_p)$ when K is a p-adic field (rather than a number field). In that setting, the role of scalar extension to $\mathbf{C}$ is replaced by scalar extension to a much larger type of ring (called a *p-adic period ring*). This also can be reinterpreted in terms of vector bundles over a curve, called the Fargues-Fontaine curve.

In this introduction we briefly review these perspectives based on vector bundles over curves in order to motivate our goal of exploring potential analogues in the global setting over a number field. (We focus our attention in this thesis on the base field $\mathbf{Q}$, so in effective smooth proper schemes over $\mathbf{Z}[1/N]$. Generalizations over number fields require inputs from class field theory that would take us too far afield here.)

1. Hodge structures

Let us begin by reviewing a reintepretation of Hodge theory as introduced by Simpson and Deligne. Denote by $\mathrm{FF}_{\mathbf{R}}$ (in analogy with the non-archimedean theory to be discussed below) the so-called "twistor" projective line given as the quotient of $\mathbb{P}^1_{\mathbf{C}}$ by the conjugate-linear involution $c : \mathbb{P}^1_{\mathbf{C}} \to \mathbb{P}^1_{\mathbf{C}}$ sending z to $-1/\overline{z}$. This form of $\mathbb{P}^1_{\mathbf{R}}$ is the unique nontrivial Brauer-Severi curve over $\mathbf{R}$.

Here are some basic features:

(1) $\mathrm{FF}_{\mathbf{R}}$ is an algebraic curve of genus zero over $\mathbf{R}$ with no $\mathbf{R}$-points. The residue field at every closed point is non-canonically isomorphic to $\mathbf{C}$ (two properties that characterize $\mathrm{FF}_{\mathbf{R}}$ up to isomorphism, as a Brauer-Severi variety).

(2) The map $\infty : \mathrm{Spec}(\mathbf{C}) \to \mathbb{P}^1_{\mathbf{C}}$ induces a closed immersion $\mathrm{Spec}(\mathbf{C}) \to \mathrm{FF}_{\mathbf{R}}$ which we still denote ∞. The formal completion at ∞ of FF_K is the formal unit disc $\mathrm{Spf}\ \mathbf{C}[\![u]\!]$.

(3) There is an evident equivalence of categories between G-equivariant vector bundles on $\mathrm{FF}_{\mathbf{R}}$ and finite dimensional $\mathbf{R}$-vector spaces. It can be promoted to an equivalence between the categories of G-equivariant semistable vector bundles of slope $w/2$ on $\mathrm{FF}_{\mathbf{R}}$ and pure Hodge structures of weight w.

In this context, one can associate, for X a smooth proper scheme over $\mathbf{C}$, a vector bundle $\mathscr{E}$ on $\mathrm{FF}_{\mathbf{R}}$ to the singular cohomology $H^i(X(\mathbf{C}), \mathbf{R})$, the following way. Define $\mathscr{E}$ to be the maximal sub-bundle of $H^i(X(\mathbf{C}), \mathbf{R}) \otimes_{\mathbf{R}} \mathscr{O}_{\mathrm{FF}_{\mathbf{R}}}$ generated by G-equivariant global sections.

One can check that $\mathscr{E}$ satisfies the following:

(1) it has global sections over $\Delta := \mathrm{Spf}\ \mathbf{C}[\![u]\!] \to \mathrm{FF}_{\mathbf{R}}$ given by the i-th cohomology of the complex:

$$\mathscr{O}_X[\![u]\!] \to u^{-1}\Omega^1_X[\![u]\!] \to u^{-2}\Omega^2_X[\![u]\!] \to \cdots$$

and, more precisely, it is the trivial vector bundle

$$H^i_{\mathrm{dR}}(X) \otimes_{\mathbf{C}} \mathscr{O}_{\Delta}$$

on Δ.

(2) it is isomorphic to the trivial vector bundle $H^i(X(\mathbf{C}), \mathbf{R}) \otimes_{\mathbf{R}} \mathscr{O}_{\mathrm{FF}_{\mathbf{R}}}$ on $\mathrm{FF}_{\mathbf{R}} - \{\infty\}$.

(3) the transition map on the overlap Δ^* is given by the comparison isomorphism:

$$\alpha_{\mathrm{dR}} : H^i_{\mathrm{dR}}(X) \otimes_{\mathbf{C}} \mathbf{C}(\!(u)\!) \simeq H^i_{\mathrm{sing}}(X, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C}(\!(u)\!)$$

induced by the Poincaré Lemma in holomorphic de Rham cohomology, tensored by $\mathbf{C}(\!(u)\!)$.

Fiberwise, $\mathscr{E}$ has fiber non-canonically isomorphic to $H^i(X(\mathbf{C}), \mathbf{C})$ at every closed point of $\mathrm{FF}_{\mathbf{R}} - \{\infty\}$ (since every such point has residue field non-canonically isomorphic to $\mathbf{C}$), and isomorphic to the Hodge cohomology

$$H^i_{\mathrm{Hodge}}(X) := \bigoplus_{p+q=i} H^p(X(\mathbf{C}), \Omega^q_X).$$

The vector bundle $\mathscr{E}$ is G-equivariant and semistable of weight $i/2$; this is a reformulation of the fact that the Hodge filtration is split by its complex conjugate.

2. p-adic Hodge theory

Let X be a smooth and proper scheme over $\mathbf{Q}_p$ (we stick to $\mathbf{Q}_p$ rather than general p-adic fields for ease of notation in this Introduction, which anyway serves just as motivation). It was a conjecture of Grothendieck that de Rham and p-adic étale cohomology of X should contain the same information, in that one should be able to recover one from the other and conversely, and in a functorial way. Fontaine made this vision precise via his discovery of the formalism of *period rings*, especially the huge $\mathbf{Q}_p$-algebras B_{cris} and B_{dR}, the latter of which is a complete discretely valued field that contains the former.

Just as $H^*_{\mathrm{dR}}(X/\mathbf{Q}_p)$ is a filtered vector space and $H^*_{\text{ét}}(X_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_p)$ has a natural Galois action, the field B_{dR} supports a Galois action and filtration via its discrete valuation, and Fontaine conjectured that naturally as B_{dR}-vector spaces:

$$H^*_{\mathrm{dR}}(X) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}} \simeq H^*_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}$$

respecting the relevant structures. Among other interesting consequences, via properties of B_{dR} this allows one to recover $H^*_{\mathrm{dR}}(X)$ from $H^*_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$:

$$H^*_{\mathrm{dR}}(X) \simeq (H^*_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}.$$

There is an analogous comparison theorem involving crystalline cohomology of the special fiber when X is equipped with a smooth proper model over $\mathbf{Z}_p$, in which case the role of B_{dR} is replaced by B_{cris}, which in turn has a Frobenius endomorphism φ serving a role analogous to that of the Frobenius operator on the crystalline cohomology of a smooth proper scheme in characteristic p.

Fontaine's precise conjectures were proved by Faltings, and in more recent years a variety of others proofs have been found via new tools in p-adic Hodge theory. A fundamental new insight in recent years has been the theory of the *Fargues-Fontaine curve* FF_K, one for each p-adic field K, that recasts p-adic comparison isomorphisms for smooth proper K-schemes in terms of vector bundles on that curve. This exhibits striking analogies with the Deligne–Simpson picture for Hodge theory over $\mathbf{C}$ as discussed above.

The scheme FF_K is a Dedekind scheme (not of finite type over $\mathbf{Q}_p$) whose function field is $E = \mathrm{Frac}(B_{\mathrm{cris}}^{\varphi=1})$ and which admits a distinguished point ∞ for which the associated completion of E is B_{dR}. Moreover, it is a deep fact that $B_{\mathrm{cris}}^{\varphi=1}$ is a principal ideal domain, and it is a fact that

$$\mathrm{FF}_K - \{\infty\} \simeq \mathrm{Spec}(B_{\mathrm{cris}}^{\varphi=1}).$$

Thus, loosely speaking, FF_K is a "gluing" of $B_{\mathrm{cris}}^{\varphi=1}$ and the discrete valuation ring B_{dR}^+ of B_{dR}. In this way, B_{dR}^+ plays the role of a formal disc, and B_{dR} is a "formal punctured disc" akin to $\mathbf{C}(\!(u)\!)$ in the discussion of Hodge theory over $\mathbf{C}$.

From this more intuitive description we gather the expectation that a regular function f on FF_K must be regular on both $\mathrm{FF}_K - \{\infty\}$ and at ∞, which means f is in the subring of E given by:

$$B_{\mathrm{cris}}^{\varphi=1} \cap B_{\mathrm{dR}}^+ = \mathbf{Q}_p.$$

Fargues and Fontaine proved that in a variety of ways, FF_K behaves like a projective line (except it is not of finite type over its field of global functions that is $\mathbf{Q}_p$). This curve leads to a profound geometric rephrasing of p-adic Hodge Theory. In particular, using analogies with the projective line, vector bundles on FF_K can be classified similarly to vector bundles on $\mathbb{P}^1$ (in terms of rank and slopes).

If X is a smooth and proper scheme over K, there is a canonical and functorial way to associate to $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ a vector bundle $\mathscr{E}$ on FF_K, uniquely determined by the following properties:

(1) over the open $U := \mathrm{FF}_K - \{\infty\} \simeq \mathrm{Spec}(B_{\mathrm{cris}}^{\varphi=1})$, $\mathscr{E}$ is the trivial vector bundle:

$$\mathscr{E}|_U \simeq H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathscr{O}_{\mathrm{FF}_K}.$$

(2) over the "formal disc" $\Delta := (\mathrm{FF}_K)^\wedge_\infty \simeq \mathrm{Spec}(B_{\mathrm{dR}}^+)$, $\mathscr{E}$ is the trivial vector bundle:

$$\mathscr{E}|_\Delta \simeq H^i_{\mathrm{dR}}(X) \otimes_K \mathscr{O}_\Delta.$$

(3) on the overlap $\Delta^* := \mathrm{Spec}(B_{\mathrm{dR}})$ the transition map is given by α_{dR}.

The p-adic de Rham comparison theorem for a smooth proper K-scheme X can be reinterpreted as constructing a map of vector bundles on FF_K:

$$\mathscr{E} \to H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathscr{O}_{\mathrm{FF}}.$$

that exhibits $\mathscr{E}$ as a "modification at ∞" (i.e., isomorphism away from ∞) of the trivial vector bundle.

In fact one can start with the trivial vector bundle and construct $\mathscr{E}$ as the maximal sub-bundle generated by $\mathrm{Gal}(\overline{K}/K)$-equivariant global sections in a formal neighborhood of ∞, which turns out to be a reformulation of the fact that the de Rham comparison isomorphism is compatible with filtrations.

The operation $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \mapsto \mathscr{E}$ allows one to use the theory of vector bundles on FF_K to assign conditions on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ (and more generally to study p-adic Galois representations of geometric origin).

3. Global dream

We wonder whether, for smooth proper schemes X over a ring of S-integers $\mathcal{O}_{K,S}$, with K a fixed number field, there exists an analog $C_{K,S}$ of the Fargues-Fontaine curves over $\mathbf{R}$ and p-adic fields for each p, and we ask ourselves whether it is possible (after enlarging S) to associate, for any choice of complex embedding $K \subset \mathbf{C}$, to the singular cohomology $H^i_{\mathrm{sing}}(X^{\mathrm{an}}, \mathcal{O}_{K,S})$ a vector bundle $\mathscr{E}$ on $C_{K,S}$ which is a modification of the trivial vector bundle:

$$H^i_{\mathrm{sing}}(X^{\mathrm{an}}, \mathcal{O}_{K,S}) \otimes_{\mathcal{O}_{K,S}} \mathscr{O}_{C_K}$$

and whose fiber at some marked closed point $\infty \in C_{K,S}$ is, at least after rationalization, related to the Hodge cohomology $H^i_{\mathrm{Hodge}}(X) := \bigoplus_{p+q=i} H^p(X, \Omega^q_{X/K})$.

Most importantly, we wonder whether we can rephrase conditions on the rational Hodge structure on $H^i_{\mathrm{sing}}(X^{\mathrm{an}}, K)$ in terms of conditions on $\mathscr{E}$.

While this thesis does not concern the construction of such $C_{K,S}$, we do, however, in the sample case $K = \mathbf{Q}$, $\mathcal{O} = \mathbf{Z}$, propose a candidate for the conjectural completion along ∞ to the conjectural $\mathscr{E}$ associated to $H^i_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbf{Z}[1/N])$.

For $K = \mathbf{Q}$, what should the partial data determining the conjectural assignment

$$H^i(X^{\mathrm{an}}, \mathbf{Z}[1/N]) \mapsto \mathscr{E}$$

look like? One expects that:

(1) $\mathscr{E}$ is isomorphic to the trivial vector bundle $H^i(X^{\mathrm{an}}, \mathbf{Z}[1/N]) \otimes_{\mathbf{Z}[1/N]} \mathscr{O}_{C_{\mathbf{Z},\{p|N\}}}$ over the open subscheme $U := C_{\mathbf{Z},\{p|N\}} - \{\infty\}$.

(2) the formal completion of $\mathscr{E}$ along the conjectural section ∞ of $C_{\mathbf{Z},\{p|N\}}$ is an arithmetic version of the i-th cohomology of the complex from the foregoing. What, precisely?

(3) the transition map on the overlap should be given by a comparison isomorphism with

$$H^i(X^{\mathrm{an}}, \mathbf{Z}[1/N])$$

involving a period ring $\mathscr{O}_{C_{\mathbf{Z},\{p|N\}}}(\Delta \cap U)$, for $\Delta := \mathrm{Spec}(\mathscr{O}^{\wedge}_{C_{\mathbf{Z},\{p|N\}},\infty})$ the "disc" around ∞ in $C_{\mathbf{Z},\{p|N\}}$.

This thesis makes progress towards providing a candidate answer to (2).

CHAPTER 2

q-divided powers

In this chapter we introduce the notion of q-divided powers. We point out to the reader that at least two concepts exist in the literature that qualify as "q-analogs" of the more familiar notion of divided powers. The first to appear is, to the best of the author's knowledge, the one in work of A. Quiros, M. Gros and B. Le Stum [**GrLS**], where the authors investigate the existence of a number of related notions of q-divided powers.

We denote by $[n]_q$ the polynomial in q with integer coefficients defined as the ratio $(q^n-1)/(q-1)$, usually called *n-th Gaussian integer* (since $[n]_q \equiv n \bmod q-1$). We further denote by $[n]_q!$ the obvious "Gaussian analog" of the factorial $n!$, by declaring $[n]_q!$ to be the product of each $[i]_q$ for $1 \leq i \leq n$.

Their q-divided powers are loosely modeled around the operations "$x^n/[n]_q!$". The authors set up their theory in a non-axiomatic way though, i.e. they do not develop axioms defining their q-divided power operations, in the spirit of the classical divided powers as in [**Berthelot**, Ch. I, §1 Def. 1.1]. They often assume their $\mathbf{Z}[q]$-algebras of interest are $[n]_q$-torsion free for every integer $n \geq 1$, in which case assigning the notion of q-divided powers of an element, and variants thereof, is a more amenable task. They discuss a number of related notions, such as q-divided power polynomial algebras in one indeterminate and several variants, as well as variations on the notion of q-derivations. The goal of their work does not appear directly related to de Rham and q-de Rham theories.

The second to appear is, to the best of the author's knowledge, in the work of B. Bhatt and P. Scholze on prismatic cohomology [**BhSch**]. Their work appeared around the time the author defended this thesis, and this manuscript was therefore revised so as to include references to [**BhSch**] and, especially, illustrate the differences between our different approaches to q-de Rham theory.

Before we start, we wish to spend a few words outlining why we are undertaking the seemingly overkill endeavor of developing a theory through axioms, in the spirit used in [**Berthelot**, Ch. I]. One may decide to declare for flat $\mathbf{Z}[q]$-algebras A that an ideal $I \subset A$ "carries q-divided powers" exactly when for every $x \in I$, we have:

$$x^n \in [n]_q! I$$

for every $n \geq 1$. In this case, the ideal generated in the $\mathbf{Z}$-flat $A/(q-1)A$ by the image of I carries divided powers in the classical sense.

In our theory, q-divided power structures will be defined without flatness assumptions and in the key cases of "q-divided power envelopes" for smooth algebras we'll be able to make contact with classical divided power envelopes (and ultimately use theorems from the crystalline theory as input to bootstrap to results with q-divided powers in cohomological settings). The notion of "q-divided power envelope", to be introduced here, will enable us to further construct a Čech-Alexander complex for a large class of $\mathbf{Z}[q]$-algebras, together with a number of its completions, which will produce cohomology theory to be compared in favorable situations to the q-de Rham theory. We will repackage the resulting completed Čech-Alexander complexs as derived global sections of a structure sheaf on an appropriate q-crystalline site, and our q-divided powers will enter the construction in much the same way as classical divided powers do for Grothendieck's crystalline site.

In other words and to summarize, q-divided power envelopes and their properties (with particular emphasis on relating their reduction modulo $q-1$ with classical divided power envelopes from the crystalline theory in nice circumstances) are the main point of this chapter, and one of the key points of the entire theory. That develop an axiomatic theory of q-divided powers has the purpose of constructing category of q-divided power algebras with which we can perform universal constructions.

This theory of q-divided powers shares a number of results that one proves for classical divided powers and divided power rings and algebras. At the same time, it also possesses a few stark differences with the classical theory.

After recalling the definition and basic features of Gauss' q-analogs $[n]_q$, for integers $n \geq 1$, and various q-analogs of familiar notions, such as q-factorials $[n]_q!$, we proceed axiomatically, to model the existence of elements "$x^n/[n]_q!$" for every integer $n \geq 1$.

1. q-divided power structures

For the benefit of the reader, we recall the classical notion of divided power structures.

Definition 1.1 Let A be a commutative ring, $I \subset A$ an ideal. By *divided powers on I* we mean a collection of maps $\gamma_n : I \to I$ for every integer $n \geq 0$, such that:

(i) For all $x \in I$, $\gamma_0(x) = 1$, $\gamma_1(x) = x$, $\gamma_n(x) \in I$ if $n \geq 1$.

(ii) For $x, y \in I$, $\gamma_n(x+y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y)$.

(iii) For $a \in A$, $x \in I$, $\gamma_n(ax) = a^n\gamma_n(x)$.

(iv) For $x \in I$, $\gamma_i(x) \cdot \gamma_j(x) = \binom{i+j}{i}\gamma_{i+j}(x)$ for all $i, j \geq 0$.

(v) For $x \in I$, $\gamma_i(\gamma_j(x)) = C_{ij} \cdot \gamma_{ij}(x)$, where $C_{ij} := \dfrac{(ij)!}{i! \cdot j!^i}$.

One observes that an easy induction shows

$$C_{ij} = \prod_{t=1}^{i-1} \binom{tj+j-1}{tj}, \tag{1.0.1}$$

and hence C_{ij} is an integer.

Definition 1.1 is [**Berthelot**, Def. 1.1], and the reader may consult Chapter I in loc.cit. for a comprehensive treatment of the classical theory.

We start with an elementary and familiar sample situation.

Example 1.2 Let R be any flat $\mathbf{Z}_{(p)}$-algebra. The ideal pR carries divided powers in the classical sense. Indeed, for any $x = pa$ in pR one can set

$$\gamma_n(x) := \frac{p^n}{n!}a^n.$$

Since the exponent of p in the prime factorization of $n!$ is $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \ldots$, one checks that indeed for every $n \geq 0$, $\gamma_n(pR) \subset pR$. That the family of maps $\{\gamma_n : pR \to pR\}_{n\geq 0}$ satisfies the axioms in Definition 1.1 is an easy verification.

Before diving into the general theory of q-divided powers, we digress on Gaussian analogs enough to be able to produce what shall deserve to be called a "q-deformation" of Example 1.2.

We assign the following:

Definition 1.3 For every integer $n \geq 1$, we define the element in $\mathbf{Z}[q]$:

$$[n]_q := \frac{q^n - 1}{q - 1},$$

and we refer to it as the *q-analog of n*. Likewise, we set:

$$[n]_q! := \prod_{i=1}^{n} [i]_q!$$

and

$$\binom{n}{i}_q := \frac{[n]_q!}{[n-i]_q! \cdot [i]_q!}$$

setting $[0]_q := 1$. We refer to each as q-factorials and q-binomial coefficients.

Remark 1.4 We note that it is not entirely obvious that $\binom{n}{i}_q$ is an element of $\mathbf{Z}[q]$. This is nonetheless easy and shown in Lemma 1.4 below. The key is the identity in $\mathbf{Z}[q]$:

$$[n]_q = \prod_{1\neq d|n} \Phi_d(q),$$

where $\Phi_d(q)$ is the d-th cyclotomic polynomial. This follows from the identity in $\mathbf{Z}[q]$:

$$q^m - 1 = \prod_{e|m} \Phi_e(q)$$

and the definition of $[n]_q$, and shows that $[d]_q \mid [n]_q$ in $\mathbf{Z}[q]$ if and only if $d \mid n$ in $\mathbf{Z}$.

Note that if p is prime then $[p]_q = \Phi_(q)$ is irreducible in the UFD $\mathbf{Z}[q]$, and $\mathbf{Z}[\![q-1]\!]/(\Phi_p) \to \mathbf{Z}_p[\zeta_p]$ via $q \mapsto \zeta_p - 1$ is an isomorphism (using that $\Phi_p(1) = p$). In particular, $[p]_q$ generates a height-1 prime in the 2-dimensional regular domain $\mathbf{Z}[\![q-1]\!]$.

Lemma 1.5 *We have in $\mathbf{Z}[q]$ the following identities:*

(i) *for all $n \geq 0$, $[n]_q! = \prod_{1\neq d\leq n} \Phi_d(q)^{\lfloor n/d \rfloor}$.*

(ii) *for all* $n, i \geq 0$*:*

$$\binom{n}{i}_q = \prod_{\lfloor n/d \rfloor > \lfloor i/d \rfloor + \lfloor (n-i)/d \rfloor} \Phi_d(q)^{\lfloor n/d \rfloor - \lfloor i/d \rfloor - \lfloor (n-i)/d \rfloor}.$$

PROOF. (i): We write:

$$[n]_q! = \prod_{i=1}^{n} [i]_q = \prod_{i=1}^{n} \prod_{1 \neq d | i} \Phi_d(q).$$

We then observe that the set of positive integers $i \leq n$ such that d divides i has size $\lfloor n/d \rfloor$, which enables us to regroup the product into $[n]_q! = \prod_{1 \neq d \leq n} \Phi_d(q)^{\lfloor n/d \rfloor}$, as desired. (ii): follows from (i) (note that the inequality condition on d forces $d \neq 1$). □

We now discuss a "q-deformation" of Example 1.2:

Example 1.6 Let p be a prime number. Let A be the $\mathbf{Z}[q]$-algebra given by the completion of the discrete valuation ring $\mathbf{Z}[\![q-1]\!]_{([p]_q)}$ (with residue field $\mathbf{Q}_p(\zeta_p)$ since $\mathbf{Z}[\![q-1]\!]/([p]_q) = \mathbf{Z}_p[\zeta_p]$).

Under the above assumption, we claim that the ideal $I = [p]_q A \subset A$ is equipped with the family of maps $\{\eta_n : I \to I\}$, indexed by the positive integers, constructed as follows. Given any $x = [p]_q a \in I$ we set, in $K := \mathrm{Frac}(A)$:

$$\eta_n(x) = \frac{[p]_q^n}{[n]_q!} a^n.$$

We now check that indeed it is the case that $\eta_n(x)$, a priori only in K, in fact always lies in I. The case $n = 1$ is clear.

The strategy consists in showing that:

(a) the ratio $[p]_q^{n-1}/\Phi_p(q)^{\lfloor n/p \rfloor}$ in K is an element of A;

(b) the ratio $[n]_q!/\Phi_p(q)^{\lfloor n/p \rfloor}$ in K is a unit in A.

If (a) and (b) are true, then we can divide in A the element $[p]_q^{n-1}/\Phi_p(q)^{\lfloor n/p \rfloor}$ by the unit $[n]_q!/\Phi_p(q)^{\lfloor n/p \rfloor}$, the result being, in K, the ratio $y := [p]_q^{n-1}/[n]_q!$. This will show that this ratio is in fact contained in A, and since $\eta_n(x) = [p]_q a y$, we deduce that $\eta_n(x)$ is contained in the ideal $[p]_q A$, as desired.

For (a): for $n \geq 0$, we have, by Lemma 1.4:

$$[n]_q! = \prod_{1 \neq d \leq n} \Phi_d(q)^{\lfloor n/d \rfloor}$$

On the other hand, since $[p]_q^{n-1} = \Phi_p(q)^{n-1}$, we deduce that the element $[p]_q^{n-1}/\Phi_p(q)^{\lfloor n/p \rfloor}$ of K is an element of A as long as $n - 1 \geq \lfloor n/p \rfloor$, which is of course the case for any $n \geq 1$.

For (b): we claim that $[n]_q!/\Phi_p(q)^{\lfloor n/p \rfloor}$ in K is a unit in A. To this end, by Lemma 1.4 again, it is enough to check that for every $1 \neq d \leq n$, with $p \neq d$, $\Phi_d(q)$ is a unit in A. We write $d = p^m e$ with p and e coprime (with $m \neq 1$ if $e = 1$). First suppose $m \geq 1$, so

$$\Phi_d(q) = \Phi_{pe}(q^{p^{m-1}}).$$

If $m \geq 2$ and $e > 1$ we have, upon specializing q at a primitive p-th root of unity ζ_p:

$$\Phi_d(\zeta_p) = \Phi_{pe}(1) = \Phi_e(1^p)/\Phi_e(1) = 1.$$

Thus, $\Phi_d(q)$ is a unit in A for all $d = p^m e$ with $m \geq 2$, $e > 1$, p and e coprime.

Now suppose $d = pe$, with $e > 1$. Then:

$$\Phi_e(q) \cdot \Phi_d(q) = \Phi_e(q^p)$$

in $\mathbf{Z}[q]$. In order to check that $\Phi_d(q)$ is a unit in A, it is enough to check that $\Phi_e(q^p)$ is a unit in A. Again after specializing q at ζ_p we get $\Phi_e(\zeta_p^p) = \Phi_e(1) \neq 0$.

Now let $d = e$ coprime to p and $d > 1$. Since ζ_p is not a root of $\Phi_d(q)$, we conclude in this case too that $\Phi_d(q)$ is a unit in A.

To summarize, we have shown that for $d = p^m e$, $m \geq 0$, $e > 1$ and p and e coprime, $\Phi_d(q)$ is a unit in A. We also know that $\Phi_{p^m}(q)$ is a unit for $m > 1$, by means of the same reasoning, since $\Phi_{p^m}(\zeta_p) = \Phi_{p^m}(1) = p \neq 0$ in $A/[p]_q$.

Since d cannot be 1 by Lemma 1.5, we are through. Note that we have shown the identity:

$$[n]_q! = u \cdot [p]_q^{\lfloor n/p \rfloor}$$

with $u \in A^\times$.

Before introducing q-divided powers, we make a few remarks.

Remark 1.7 We note from Example 1.6 that we have shown that $[p]_q$ divides $[n]_q!$ in A a total of $\lfloor n/p \rfloor$ times. This is much less than the times p divides $n!$, which is $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \cdots$. It is however still enough to ensure that the ratio $[p]_q^{n-1}/[n]_q!$ in K is an element of A. We further observe that the arguments in Example 1.6 have never really used completeness of A in the $(q-1, [p]_q)$-adic topology, which was only assumed for psychological reasons. The example can be restated verbatim for $A = \mathbf{Z}[q]_{\mathfrak{p}}$, with $\mathfrak{p}$ the prime ideal generated by $[p]_q$, and $I = \mathfrak{p}$.

Remark 1.8 Keeping A as in Example 1.6, if n is a power of p, say p^r for $r \geq 1$, one can be more precise.

First, we observe that for $I = [p]_q A$ and $x = [p]_q a \in I$, we have:

$$\eta_p(x) = \frac{[p]_q^p}{[p]_q \cdot [p-1]_q!} a^p = u \cdot [p]_q^{p-1} a^p$$

where u is a unit. We then further observe (using $[i]_{q^p}[p]_q = [pi]_q$) that we have:

$$[p^2]_q! = \widetilde{u} \cdot \prod_{i=1}^{p} [i]_{q^p} \cdot [p]_q^p$$

for $\widetilde{u} \in A^\times$ (since $[d]_q \in A^\times$ with $p \nmid d$). Now note that the map $\phi : A \to A$ sending $q \mapsto q^p$ is a ring homomorphism, and hence if $[i]_q$ is a unit in A, so is $\phi([i]_q)$. This latter is $[i]_{q^p}$. It follows that $[p^2]_q! = v \cdot [p]_{q^p} \cdot [p]_q^p$, with v a unit in A. Moreover, $[p]_{q^p}$ is a unit in A since $[p]_{q^p} = \Phi_{p^2}$ has nonzero value on ζ_p. Thus, $\eta_{p^2}([p]_q a)$ is an $A^\times$-multiple of $[p]_q^{p^2-p-1}[(p)_q a^p)$. This gives another proof that $\eta_{p^2}(x) \in I$.

We can likewise show that $\eta_{p^r}(x)$ is in I in a similar way, by induction on r. Indeed, we have:

$$[p^r]_q! = u \cdot \prod_{i=1}^{p^{r-1}} [i]_{q^p} \cdot [p]_q^{p^{r-1}}$$

for $u \in A^\times$. Since $[i]_{q^p} = \prod_{1 \neq d | i} \Phi_d(q^p)$, by reducing modulo $[p]_q$ (which amounts to specializing q at a primitive p-th root of unity ζ_p) we get:

$$[i]_{q^p}(\zeta_p) = \prod_{1 \neq d | i} \Phi_d(1) \neq 0.$$

As a result, we have $[p^r]_q! = v \cdot [p]_q^{p^{r-1}}$ for $v \in A^\times$, so for $r \geq 2$ we have that $\eta_{p^r}([p]_{q^p})$ is a unit multiple of $[p]_q^{p^r - p^{r-1} - 1}([p]_q q^{p^r})$.

Remark 1.9 For A again as in Example 1.6, we note how, when $p = 2$, $\eta_4(x)$ and $\eta_2(\eta_2(x))$ are related in K by the element:

$$\frac{[4]_q!}{[2]_q! \cdot [2]_q!^2}$$

but this element is *not* in A, let alone in $\mathbf{Z}[q]$. Indeed, it is $(q^2+1)/(q+1)^2$. The moral is that even though in A one has access to a wider supply of units than in $\mathbf{Z}[q]$, and so nontrivial relations such as the ones in the foregoing become true, it is still the case that basic elements such as the one above are still not in A anyway. This makes the task of singling out a "q-analog" of axiom (v) in Definition 1.1, somewhat delicate.

Remark 1.10 We point out a final aspect of Example 1.6, for which we consider the following diagram:

$$\begin{array}{ccccc}
([p]_q) & \xrightarrow{\subset} & \mathbf{Z}[\![q-1]\!]^\wedge_{([p]_q)} & \xrightarrow{q \mapsto \zeta_p} & \mathbf{Z}_p[\zeta_p] \\
\downarrow & & \downarrow{\scriptstyle q \mapsto 1} & & \downarrow{\scriptstyle \zeta_p \mapsto 1} \\
(p) & \xrightarrow[\subset]{} & \mathbf{Z}_p & \xrightarrow[p \mapsto 0]{} & \mathbf{F}_p
\end{array}$$

By the argument given in Example 1.6, and adopting the same notation, we see that under the reduction modulo $q-1$ map $A \to \mathbf{Z}_p$, the maps $\eta_n : [p]_q A \to [p]_q A$ are turned into the maps $\gamma_n : p\mathbf{Z}_p \to p\mathbf{Z}_p$ of Example 1.2.

More precisely, for any $x \in [p]_q A$, with image $\bar{x}$ in $\mathbf{Z}_p$ under the quotient $A \to \mathbf{Z}_p$, the image of $\eta_n(x)$ agrees with $\gamma_n(\bar{x})$. To see this, we use that for every integer $n \geq 1$, A is $[n]_q$-torsion free and $\mathbf{Z}_p$ is n-torsion free. Indeed, Example 1.6 shows that for every such x, $x^n \in [n]_q! I$. This implies that $\bar{x}^n$ is in the ideal of $\mathbf{Z}_p$ given by $n!\bar{I}$, where $\bar{I}$ is the ideal of $\mathbf{Z}_p$ generated by the image of $[p]_q$ under $A \to \mathbf{Z}_p$, which is of course $p\mathbf{Z}_p$. By torsion-freeness, there can be only one element y in $\mathbf{Z}_p$ such that $n! \cdot y = \bar{x}^n$. Both $\gamma_n(\bar{x})$ and the image of $\eta_n(x)$ in A satisfy the relation that y satisfies, whence these agree. It also means that the image of $\eta_n(x)$ in $\mathbf{Z}_p$ under reduction modulo $q-1$ does not depend on the choice of x, but only on $\bar{x}$.

While Example 1.6 is quite elementary to begin with, it does offer a few instructive takeaways. Namely, for A a $\mathbf{Z}[q]$-algebra and $I \subset A$ an ideal:

(a) Whichever the final definition of q-divided powers of $x \in I$ shall be, it is a reasonable expectation that, in the special case where for each integer $n \geq 1$ both A is $[n]_q$-torsion free and $A/(q-1)A$ is n-torsion free, x admits an "n-th q-divided power" precisely when $x^n \in [n]_q!I$.

(b) Specialization of q at 1 should "turn" q-divided powers into divided powers in the sense of Definition 1.1.

(c) The ideal $([p]_q)$ ofthe discrete valuation ring $\mathbf{Z}[\![q-1]\!]^{\wedge}_{([p]_q)}$ should be treated as the correct analog of the ideal (p) of $\mathbf{Z}_p$ in the classical theory.

There are of course a number of more serious reasons why we shall assign the definition we give later in this chapter for the concept of q-divided powers. However, each of the points above already draws a distinction between our theory and that in [**BhSch**, §16].

We note that in the above Example 1.6, the elements $\eta_n(x)$ for $x \in I$ are a replacement for "$x^n/[n]_q!$". There are other reasons why modeling q-divided powers around the operations "$x^n/[n]_q!$" is a fruitful way forward, as we shall see.

For the moment, the takeaway of the above motivational observations is that in the sequel, we shall not regard the ideal $(q-1) \subset \mathbf{Z}[q]$ as an analog of the ideal $(p) \subset \mathbf{Z}_{(p)}$ from the classical theory. In particular, we shall not expect the ideal $(q-1)$ to carry "q-divided powers", and in fact it typically will not, in almost every situation.

Before we start discussing the notion of q-divided powers and q-divided power rings and algebras, we spend some time pinning down some crucial differences of our approach to q-crystalline cohomology,from the construction in [**BhSch**, §16], with particular emphasis on why a p-locality assumption (which we have bypassed) seems to be essential in [**BhSch**].

As mentioned above, the map $\eta_n : I \to I$ in Example 1.6 has the role of replacing the operation $x^n/[n]_q!$ on I in very much the same way as γ_n in a general divided power structure replaces $x^n/n!$ on $(p) \subset \mathbf{Z}_{(p)}$.

Set $A = \mathbf{Z}[\![q-1]\!]_{\mathfrak{p}}$, with $\mathfrak{p} = [p]_q A$. The ideal $I \subset \mathfrak{p}$ of A equipped with the data of $\{\eta_n\}$ is for us the basic instance of a "q-divided power structure", a concept which we introduce below, and deserves to be regarded as a "q-deformation" of the ideal $(p) \subset \mathbf{Z}_{(p)}$ equipped with its divided power structure γ, as we have discussed in Remark 1.10.

In our quest for an axiomatic theory of q-divided powers replacing the operation "$x^n/[n]_q!$" for every $n \geq 1$, we immediately hit a snag: whatever such a theory will be, it will lack a usable analogue of the binomial theorem. Indeed, for any $\mathbf{Z}[q]$-algebra A and $x, y \in A$, the best analogue of the binomial theorem that we know of is

$$\prod_{i=0}^{n-1}(x + q^i y) = \sum_{i=0}^{n} q^{(n-i)(n-i-1)/2} \binom{n}{i}_q x^i y^{n-i} \tag{1.0.2}$$

(proved by induction on n), yet this is clearly not symmetric in x and y, and the left side has little to do with the expression $(x+y)^n/[n]_q!$.

In other words, upon axiomatizing a q-analogue of divided powers, we expect to lose the ability to check that an ideal carries such a structure by checking it on generators (apart from the case the ideal is principal), an ability that axiom (ii) in Definition 1.1 gives us in the classical theory. This will require us to run certain arguments in a more coordinate-free way (e.g., the construction of q-divided power envelopes is by necessity more abstract that in the classical case, even for smooth algebras). Apart from this technical difficulty, the theory will work similarly to its classical counterpart that will be recovered by specializing q to be 1. It will be essential to us to relate q-divided power envelopes to classical divided power envelopes. As it turns out, an advantage of setting up the theory axiomatically is that in favorable circumstances the reduction modulo $(q-1)$ of a q-divided envelope is closedly related to a classical divided power envelope (with a discrepancy that will be harmless for our cohomological goals).

In [**BhSch**, §16] the authors take quite a different approach. Suppose D is a $(p, q-1)$-adically complete $\mathbf{Z}_p[\![q-1]\!]$-algebra. The approach in loc.cit. seems to be dictated by the desire to endow the ideal $(q-1) \subset D$ with an analogue of a divided power structure, which we recall is somewhat orthogonal to our approach, since $(q-1)$ will typically not carry such structure in our theory, while $[p]_q D$ would, under suitable assumptions on D.

In order to set up a theory for which the ideal $(q-1)D$ carries such structure, the authors equip themselves with the additional datum of "Frobenius lifts" on D (in the p-torsion free setting this amounts to a single operator whose reduction modulo p agrees with the p-th power map). The idea to use a collection of "Frobenius lifts" was first introduced by Pridham in [**Prid**], via the observation that the λ-structure on $\mathbf{Z}[\![q-1]\!]$ assigned by requiring $\lambda(q) = 0$, or equivalently by declaring the map $q \mapsto q^p$ to be the pth Adams operation for every prime p, enables one to define, for every positive integer n, a map γ_n on $(q-1)$ as

$$\gamma_n(x) := \frac{\phi_n(x)}{[n]_q} - \frac{\phi_n(x) - x^n}{n}$$

where ϕ_n is the n-th Adams operation. For λ-, δ- and Adams operations, and the relevant theory, we refer the reader to [**Joyal**].

Indeed, by definition of the q-analogue of an integer, $\phi_n(q-1) = q^n - 1$ is in the ideal generated by $[n]_q$, and so for any $x \in (q-1)$ the ratio $\phi_n(x)/[n]_q$ lies in $\mathbf{Z}[\![q-1]\!]$, so γ_n makes sense as an operation on the ideal $(q-1)$ valued in $\mathbf{Z}[\![q-1]\!]$. Note that γ_n is a q-deformation of x^n/n. For $n = p$ the discrepancy between x^n/n and $x^n/n!$ is harmless since it is a unit $(p-1)! \in \mathbf{Z}_{(p)}$.

The work in [**BhSch**] sets up an analogue of the crystalline site by building on this observation and works with a concept of "q-divided power thickening" using ideals carrying analogues of the above maps γ_n. With this approach one doesn't quite get a q-deformation of classical divided powers, though, and it is here that the authors appear to be bound to require all the $\mathbf{Z}[q]$-algebras entering their construction to be at least "p-local", usually p-adically complete.

Indeed, upon specializing the map γ_n at $q = 1$, we obtain a collection of maps that gives a replacement for the operation "x^n/n" rather than "$x^n/n!$". One notes that in order for the crystalline site and cohomology to work, i.e. in order to have a Poincaré Lemma for affine spaces that enables one to compute cohomology using appropriate complexes of differential forms, which is usually a way to establish good properties of such cohomology, one actually needs divided power structures as in

Definition 1.1 in the definition of divided power rings, algebras and thickenings. This weakening will not yield an exact augmented de Rham complex in the respective theory: roughly, adjoining "f^n/n" for every n and every f in some ideal in a polynomial $\mathbf{Z}[q]$-algebra will render some cycles in the resulting de Rham complex which weren't integrable before, integrable, while, however, introducing new non-integrable cycles, such that f^n/n itself.

To bypass this problem (the lack of the ability to recover usual divided power structures on the nose upon specializing q at 1), of course one may simply further assume all $\mathbf{Z}[\![q-1]\!]$-algebras involved are p-adically complete and separated, and then the datum of γ_p above is indeed sufficient to construct arbitrary divided powers after specializing q at 1. This is what is done in [**BhSch**].

The work [**BhSch**] equips every pair (D, I), for D a $(p, q-1)$-adically (separated and) complete $\mathbf{Z}_p[\![q-1]\!]$-algebra and $I \subset D$ a $(p, q-1)$-adically complete ideal of D containing $(q-1)$, with the additional datum of a δ-*structure* on D in the sense of Joyal (see [**Joyal**]). Maps between such pairs are required to be compatible with the δ-structures. Additional conditions are introduced to define what it means for I to carry a q-deformation of divided powers (see [**BhSch**, §16, Def. 16.2]), which essentially amounts to requiring, first, that for every $x \in I$ we have $\phi_p(x) \in [p]_q D$ for every $x \in I$. Here ϕ_p is the Frobenius lift on D determined by the δ-structure; morally, this is akin requiring that "$x^p/[p]_q$" is defined in D for every element x of I. This enables one to define the map:

$$\gamma_p(x) := \frac{\phi_p(x)}{[p]_q} - \delta(x),$$

as a map $\gamma_p : I \to D$. The authors then declare that I carries a q-divided power structure if, in addition, $\gamma_n(I)$ is contained in I.

We highlight again that use of δ-structures in [**BhSch**] appears to be motivated by the aim of building on the intuition that the ideal $(q-1)$ should carry "q-divided powers". In turn, if one is to recover usual divided powers after specializing q at 1 and to have any hope that the resulting q-crystalline cohomology compares correctly with de Rham cohomology, then one needs D to at least satisfy some p-locality condition, in order to reconstruct all divided powers on $D/(q-1)$ out of "x^p/p" alone. p-adic completeness of D appears to be needed throughout [**BhSch**] for a number of other reasons too, which we refrain from discussing here.

The task of constructing a "q-crystalline site" that makes sense for $\mathbf{Z}[q]$-algebras, or at least $(q-1)$-adically separated and complete $\mathbf{Z}[\![q-1]\!]$-algebras, with no p-adic completeness, appears to therefore be tied to the task of getting rid of δ-structures altogether. It seems reasonable to expect that it is due to the necessity of δ-structures in the theory from [**BhSch**], in turn given by the desire of endowing the basic ideal $(q-1)D \subset D$ with "q-divided powers", that the authors in loc.cit. confine themselves into a p-local, and often p-adically separated and complete, situation.

It appears to the author that there is not much of a choice but to proceed as in the current literature, mostly around quantum groups and Lie algebras, by defining q-divided powers by axiomatizing them as a replacement for the operation $x^n/[n]_q!$. In so doing, we have to accept that the ideal $(q-1)$ generally does not carry a q-divided power structure, even in the p-adically complete case. Before introducing our notion of q-divided powers, we first discuss a preliminary concept: that of a "coarse"

q-divided power structure. This concept is modeled on the case of $\mathbf{Z}[q]$-algebras A that are $[n]_q$-torsion free for each integer $n \geq 1$ (not necessarily $\mathbf{Z}$-flat). Later q-divided powers will be defined to be coarse q-divided powers satisfying some additional conditions.

We assign the following (omitting analogue of axioms (ii) and (v) from Definition 1.1).

Definition 1.11 Let A be a $\mathbf{Z}[q]$-algebra and I an ideal of A. By *q-divided powers on I* we mean a collection of maps $\eta_n : I \to I$ for every integer $n \geq 0$, such that:

(i) For all $x \in I$, $\eta_0(x) = 1$, $\eta_1(x) = x$, $\eta_n(x) \in I$ if $n \geq 1$.

(iii) For $a \in A$, $x \in I$, $\eta_n(ax) = a^n \eta_n(x)$.

(iv) For $x \in I$, $\eta_i(x) \cdot \eta_j(x) = \binom{i+j}{i}_q \eta_{i+j}(x)$ for all $i, j \geq 0$.

Remark 1.12 For $n \geq 1$ we have $\eta_n(0) = 0$ for any ideal I of A and any q-divided power structure η on I. (This follows from axiom (iii) in Definition 1.11, applied to $a = 0$.)

Notation 1.13 For a $\mathbf{Z}[q]$-algebra A, an ideal $I \subset A$ and coarse q-divided powers $\{\eta_n : I \to I\}_{n \geq 0}$, we denote the collection of the maps η_n by η, to abbreviate. We call it a *q-divided power structure* on I.

The following Lemma provides a large class of examples of coarse q-divided power structures. It also shows that $\eta_n(x)$ is a replacement for the operation "$x^n/[n]_q!$" on I. It further shows that in the special situation in which A is $[n]_q$-torsion free for every $n \geq 1$, coarse q-divided power structures are a more intrinsic notion.

Lemma 1.14 *Let A be a $\mathbf{Z}[q]$-algebra, I an ideal of A, and η a q-divided power structure on I.*

(1) *We have $[n]_q! \eta_n(x) = x^n$ for $n \geq 1$, $x \in I$.*

(2) *For any $x, y \in I$, $\prod_{i=0}^{n-1}(x + q^i y) = [n]_q! \sum_{j=0}^{n} q^{(n-j)(n-j-1)/2} \eta_j(x) \eta_{n-j}(y)$.*

Assume A is $[n]_q$-torsion free for every positive integer n as a $\mathbf{Z}[q]$-module. Then:

(3) *a q-divided power structure on I, if it exists, is unique.*

(4) *If, for $n \geq 0$, $\eta_n : I \to I$ is a family of maps, then it is a q-divided power structure if and only if $[n]_q! \eta_n(x) = x^n$ for all $x \in I, n \geq 0$.*

Proof. For (1): Suppose I carries q-divided powers. Then by iterating axiom (iv) applied to $i = n$ and $j = 1$ in Definition 1.11 we obtain $x^n = [n]_q! \eta_n(x)$ for every integer $n \geq 1$. This implies $x^n \in [n]_q! I$ for every integer $n \geq 1$ and every $x \in I$.

For (2): we "divide and multiply" by $[j]_q!$ and $[n-j]_q!$ (which makes sense since their product divides $[n]_q!$ in $\mathbf{Z}[q]$) to obtain

$$\sum_{j=0}^{n} q^{(n-j)(n-j-1)/2} \binom{n}{j}_q ([j]_q! \eta_j(x))([n-j]_q! \eta_{n-j}(y)) = \sum_{j=0}^{n} q^{(n-j)(n-j-1)/2} \binom{n}{j}_q x^j y^{n-j}.$$

But it is an easy induction on n that this final expression is equal to $\prod_{i=0}^{n-1}(x + q^i y)$ (see (1.0.2))

For (3): it follows from (1) and $[n]_q$-torsion freeness of A for every integer $n \geq 1$.

For (4): Assume that for every integer $n \geq 1$, we have $x^n \in [n]_q!I$ for every $x \in I$. Then we set $\eta_0(x) = 1$ and $\eta_1(x) = 1$, as well as $\eta_n(x)$ to be the unique element a_n of I such that $[n]_q!a_n = x^n$. That such element exists is due to the assumption on x. Uniqueness of such an element is due to the $[n]_q$-torsion freeness assumption on A for every $n \geq 1$. The η_n's are, therefore, well defined, and we only need to check that they indeed satisfy the axioms in Definition 1.11.

Axiom (i) is clear by design of the η_n's. So is axiom (iii), since $[n]_q!(\eta_n(ax) - a^n\eta_n(x)) = 0$ and A is $[n]_q!$-torsion free. For axiom (iv), we have:

$$\begin{aligned}\binom{i+j}{i}_q \cdot [i]_q! \cdot [j]_q! \cdot \eta_i(x)\eta_j(x) &= \binom{i+j}{i}_q x^i \cdot x^j \\ &= \binom{i+j}{i}_q x^{i+j} \\ &= \binom{i+j}{i}_q [i+j]_q!\eta_{i+j}(x)\end{aligned}$$

and since $\binom{i+j}{i}_q \cdot [i]_q! \cdot [j]_q! = [i+j]_q!$, we can cancel out $[i+j]_q!$ on both sides since A is $[i+j]_q!$-torsion free, to conclude. □

As we have just discussed a criterion to decide when an ideal I in a $\mathbf{Z}[q]$-algebra A carries q-divided powers in the case A is $[n]_q$-torsion free, we spend a few more words to further note that [**BhSch**] focuses on the situation where their q-divided power pairs (D, I) have $[p]_q$-torsion free D. This may be necessary for the authors to avoid an axiomatization of the features of their map γ_p on I, and is sufficient to establish their main results.

In [**BhSch**], the authors prove that the reduction modulo $q-1$ of their q-divided power envelop indeed is the classical (p-adically separated and complete) divided power envelop in the case of algebras that are p-adically (ind-)étale over a free algebra and the ideal is generated by a p-adically regular sequence. This is enough for their main applications. In our approach, we will likewise establish a close connection with a classical divided power envelope in favorable cases that suffice for our intended applications.

Example 1.15 The maps η_n in Example 1.6 are a q-divided power structure on the ideal $I = ([p]_q)$. Indeed, the arguments in the example show that for any $x \in I$ and any integer $n \geq 1$, $x^n \in [n]_q!I$. Since the $\mathbf{Z}[q]$-algebra A in that example is $[n]_q$-torsion free for every $n \geq 1$, we may apply Lemma 1.14(4), to conclude. Of course, the maps η_n are not merely q-divided powers: their "specialization at $q = 1$", as described in Remark 1.10, is a divided power structure on the specialization of the ideal I in $A/(q-1)A$. We saw this was a consequence of the fact that, in addition to being $[n]_q$-torsion free for every integer $n \geq 1$, $A/(q-1)A$ is also n-torsion free for every such integer. This additional $\mathbf{Z}$-flatness might possibly fail for more general A that are not $[n]_q$-torsion free for every $n \geq 1$.

Lemma 1.16 *Let A be a $\mathbf{Z}[q]$-algebra, $I \subset A$ an ideal. Assume that A is f-torsion free for any $f \in \{q-1\} \cup \{[n]_q, n \geq 1\}$, and that $A/(q-1)A$ is bZ-flat. Then for a q-divided power structure on I, there is a unique compatible divided power structure on the image $\overline{I}$ of I in $A/(q-1)A$.*

Proof. By Lemma 1.14, I carries a q-divided power structure if and only if for every $x \in I$, we have $x^n \in [n]_q!I$. Then every element $\bar{x}$ of $\overline{I}$ satisfies: $\bar{x}^n \in n!\overline{I}$. This implies that $\overline{I}$ carries divided powers γ_n in the classical sense since $A/(q-1)A$ is $\mathbf{Z}$-flat by hypothesis. For any $x \in I$, the image of $\eta_n(x)$ in $\overline{I}$ serves as $\gamma_n(\bar{x})$ for $\bar{x}$ the image of x in $\overline{I}$ since $[n]_q!\eta_n(x) = x^n$ and $A/(q-1)A$ is $\mathbf{Z}$-flat. □

1.1. Morphisms involving q-divided power structures. We now give some more basic definitions.

Definition 1.17 A *q-pair* is a pair (A, I) where A is a $\mathbf{Z}[q]$-algebra, $I \subset A$ is an ideal. Morphisms of pairs $(A, I) \to (B, J)$ are morphisms $f : A \to B$ of $\mathbf{Z}[q]$-algebras such that $f(I) \subset J$. We denote by qPairs the resulting *category of q-pairs.*

For a q-pair (A, I), an (A, I)-algebra (B, J) is a map of pairs $(A, I) \to (B, J)$. Morphisms of (A, I)-algebras are morphisms of pairs over (A, I). We denote by (A, I)-Alg the category of (A, I)-algebras.

Definition 1.18 A *q-divided power ring* is a triple (A, I, η) where (A, I) is a q-pair and η is a q-divided power structure on I. Morphisms of q-divided power rings $(A, I, \eta) \to (A', I', \eta')$ are morphisms of pairs $f : (A, I) \to (B, J)$ such that, for every $x \in I$ and $n \geq 0$, $\eta'_n(f(x)) = f(\eta_n(x))$.

A *q-divided power algebra* over a q-divided power ring (A, I, η) is a divided power ring (A', I', η') equipped with a morphism of q-divided power rings $(A, I, \eta) \to (A', I', \eta')$. Morphisms of q-divided power (A, I, η)-algebras are morphisms of q-divided power rings over (A, I, η).

We remark here that since a morphism of q-divided power rings $(A, I, \eta) \to (B, J, \kappa)$ is induced by a morphism of $\mathbf{Z}[q]$-algebras $f : A \to B$, the maps $\kappa_n \circ f : I \to J$ satisfy:

(i)$_\kappa$ For all $x \in I$, $\kappa_0(f(x)) = 1$, $\kappa_1(f(x)) = f(x)$, $\kappa_n(f(x)) \in J$ if $n \geq 1$,

(iii)$_\kappa$ For $a \in A$, $x \in I$, $\kappa_n(f(a)f(x)) = f(a)^n\kappa_n(f(x))$,

(iv)$_\kappa$ For $x \in I$, $\kappa_i(f(x)) \cdot \kappa_j(f(x)) = \binom{i+j}{i}_q \kappa_{i+j}(f(x))$ for all $i, j \geq 0$,

We also have:

(i)$_\eta$ For all $x \in I$, $f(\eta_0(x)) = 1$, $f(\eta_1(x)) = f(x)$, $f(\eta_n(x)) \in J$ if $n \geq 1$.

(iii)$_\eta$ For $a \in A$, $x \in I$, $f(\eta_n(ax)) = f(a)^n f(\eta_n(x))$.

(iv)$_\eta$ For $x \in I$, $f(\eta_i(x)) \cdot f(\eta_j(x)) = \binom{i+j}{i}_q f(\eta_{i+j}(x))$ for all $i, j \geq 0$.

If, in addition, $\kappa_n \circ f = f \circ \eta_n$ on I for every $n \geq 0$, then by $\mathbf{Z}[q]$-linearity of f and by the above two lists of properties, following from the axioms of q-divided powers, we also have that the expressions in $(*)_\kappa$ and $(*)_\eta$ agree for $* =$i,iii,iv.

The notion of morphism of q-divided power rings, and hence the notion of q-divided power algebra over a q-divided ring and morphisms of such, therefore all make sense.

We point out once more that if $q = 1$ in A, a q-divided power structure on I is a weakening of the notion of divided power structure on I since analogues of the "additivity" and "composition" axioms have been removed.

2. q-divided power rings and algebras

Lemma 2.1 *Let (A, I, η) be a q-divided power ring and $q\mathrm{PD}_{/(A,I,\eta)}$ be the category of of q-divided power algebras over (A, I, η). Then $q\mathrm{PD}_{/(A,I,\eta)}$ has all limits and they agree with limits in the category of $\mathbf{Z}[q]$-algebras.*

Of course, the very last statement contains an abuse of terminology; it means that if $\{(A_s, I_s, \gamma^{(s)})\}_{s\in\Sigma}$ is a diagram in $q\mathrm{PD}_{/(A,I,\eta)}$, with Σ a small index category, and if (B, J, κ) is its limit, then $B = \lim_{s\in\Sigma} A_s$ in the category of A-algebras. However, limits in the category of A-algebras agree with limits in the category of $\mathbf{Z}[q]$-algebras.

Proof. The empty limit is the zero q-divided power ring $(0, (0), \varnothing)$. The product of a collection of q-divided power rings (A_t, I_t, γ_t), $t \in T$ is given by $(\prod A_t, \prod I_t, \kappa)$ where $\kappa_n((x_t)) = (\kappa_{t,n}(x_t))$. The equalizer of $\alpha, \beta : (B, J, \kappa) \rightrightarrows (B', J', \kappa')$ is just $C = \{x \in B \mid \alpha(x) = \beta(x)\}$ with ideal $C \cap J$ and induced q-divided powers. Of course one still needs to check that for every $n \geq 0$, $\kappa_n : C \cap J \to J$ really factors through $C \cap J$, and all the axioms of course follow from the axioms for κ. This is true because the set underlying C is the set-theoretic equalizer of α and β. It follows that all limits exist, since products and equalizers do. $\square$

We will often write (A, I) rather than (A, I, η) without risk of confusion. We will adopt the same notation for q-divided power algebras over (A, I).

The following lemma illustrates a very general category-theoretic phenomenon in the case of q-divided power algebras.

Lemma 2.2 *Let $q\mathrm{PD}_{/(A,I)}$ be the category of q-divided power algebras. Let $F : q\mathrm{PD}_{/(A,I)} \to \mathrm{Sets}$ be a (covariant) functor. Assume that*

1. *there exists a cardinal ν such that for every $f \in F(B, J)$ there exists a morphism $(B', J') \to (B, J)$ of $q\mathrm{PD}_{/(A,I)}$ such that f is the image of $f' \in F(B', J')$ and $|B'| \leq \nu$, and*
2. *F commutes with limits.*

Then F is representable; i.e., there exists an object (C, K) of $q\mathrm{PD}_{/(A,I)}$ such that

$$F(B, J) = \mathrm{Hom}_{q\mathrm{PD}_{/(A,I)}}((C, K), (B, J))$$

functorially in (B, J).

Proof. This is an application of a general representablity result [**SP**, Tage 0AHM]. We sketch the argument here, for completeness. One calls $\mathcal{C}$ the essentially small category whose objects are pairs $((B, I), f)$ with (B, I) a q-divided power algebra over (A, I) satisfying $|B| \leq \nu$, and morphisms are maps $h : (B, I) \to (B', I')$ in $q\mathrm{PD}_{/(A,I)}$ such that $F(h)(f) = f'$. We set:

$$(C', K') := \lim_{((B_j, J_j), f_j)\in\mathcal{C}} (B_j, J_j)$$

which exists by assumption. Since F commutes with limits, we find a universal element $f \in F(C', K')$ mapping to $f_j \in F(B_j, J_j)$ under F applied to the projection $(C', K') \to (B_j, J_j)$ for all j. This yields a transformation of functors:

$$\xi : \mathrm{Hom}_{q\mathrm{PD}_{/(A,I)}}((C', K'), \cdot) \to F(\cdot)$$

One checks that by assumption (1), ξ is surjective when evaluated at arbitrary objects in $q\mathrm{PD}_{/(A,I)}$.

In order to find a representing object for F, we replace (C', K') with (C, K), the equalizer of all the self maps $g : (C', K') \to (C', K')$ such that $F(g) = f$. Since F commutes with limits, and hence with equalizers, by assumption (1) again, we see that surjectivity of ξ implies surjectivity of

$$\xi' : \mathrm{Hom}_{q\mathrm{PD}_{/(A,I)}}((C, K), \cdot) \to F(\cdot).$$

It is elementary to now check ξ' is, in addition, injective when evaluated at arbitrary objects in $q\mathrm{PD}_{/(A,I)}$. ☐

For q-divided powers we cannot generally expect to make constructions by chasing "generators and relations" as in the theory of usual divided powers (e.g., there is no "addivity" axioms). The next Lemma is therefore established more abstractly, as an application of Lemma 2.2.

Proposition 2.3 *The category of q-divided power algebras over (A, I) has all colimits.*

As with usual divided powers, the forgetful functor $(B, J) \mapsto B$ does not commute with colimits.

Proof. The empty colimit is $\mathbf{Z}[q]$ as a $\mathbf{Z}[q]$-algebra, with q-divided power ideal (0). We now discuss general colimits. Let Σ be an essentially small index category and let $s \mapsto (B_s, J_s, \eta_s)$ be a diagram. Consider the functor

$$F(B, J, \eta) = \lim_{s \in \Sigma} \mathrm{Hom}((B_s, J_s, \eta_s), (B, J, \eta))$$

Note that any $f = (f_s)_{s \in \Sigma} \in F(B, J, \eta)$ has the property that the images $f_s(B_s)$ generate an A-algebra B' of B of bounded cardinality ν (as B varies) and that the images $f_s(J_s)$ generate a an ideal J' of B' contained in J. Using the functions η_n for (B, J) nearly equips (B', J') with a q-divided power structure; the problem is that the outputs of these functions may not lie in J'. So we replace B' with the A-subalgebra B'' of B generated by $\eta_n(J') \subset J$, and replace J' with the ideal $J'' \subset B''$ generated by $\eta_n(J') \subset J$.

Iterating this process countably many times brings us to a q-divided power ring with cardinarily bounded independently of (B, J), and we get a factorization of f as a f' in $F(B')$ followed by the inclusion $B' \to B$. Clearly F commutes with limits, so we may apply Lemma 2.2 to conclude that F is representable. ☐

The key application of Proposition 2.3 is the existence of universal q-divided power envelopes, which we discuss in the next section.

3. q-divided power envelopes

The construction of the following lemma will be called the *q-divided power envelope*. It will play a crucial role later insofar it will ultimately make our q-crystalline theory more tangible, in much the same way as divided power envelopes enable one to use Čech theory to compute cohomology and relate it to the cohomology of an appropriate de Rham complex via a suitable divided power Poincaré Lemma.

Lemma 3.1 *Let (A, I, η) be a q-divided power ring and consider the category $q\mathrm{PD}_{/(A,I,\eta)}$ of q-divided power algebras over (A, I, η). Let $J \subset B$ be an ideal with $IB \subset J$.*

There exists a q-divided power algebra (B', J', δ) over (A, I, η) and a map of pairs $(B, J) \to (B', J')$ over (A, I) such that for any q-divided power algebra (C, K, κ) over (A, I, η), any map of pairs $(B, J) \to (C, K)$ over (A, I) uniquely extends to a map of q-divided power algebras $(B', J', \delta) \to (C, K, \kappa)$.

In other words, the map of sets

$$\mathrm{Hom}_{q\mathrm{PD}_{/(A,I,\gamma)}}((B', J', \delta), (C, K, \kappa)) \to \mathrm{Hom}_{(A,I)\text{-Alg}}((B, J), (C, K))$$

is bijective.

Proof. Our aim is to prove the representability of the functor

$$(C, K, \kappa) \mapsto \mathrm{Hom}_{(A,I)\text{-Alg}}((B, J), (C, K)),$$

which we will do via Lemma 2.2. Condition (2) is clear due to the second assertion in Lemma 2.1. For condition (1), we proceed as follows.

Let (C, K, κ) be a q-divided power algebra over (A, I, η).

For any map of pairs $f : (B, J) \to (C, K)$ over (A, I), consider the image A-subalgebra $R_0 = f(B) \subset C$. The ideal $J_0 = f(J)R_0 \subset R_0$ may not admit q-divided powers in R_0 (in the sense of being stable under κ on K), but its elements do admit such in C since $J' \subset K$ (in the same sense as we encountered in the proof of Proposition 2.3). Hence, it makes sense to form the R_0-subalgebra $R_1 \subset C$ generated by all q-divided powers of elements of J_0. The ideal $J_1 \subset R_1$ generated by those q-divided powers may not itself admit q-divided powers in R_1 (in the sense of stability under κ, once again), but does so inside C since $J_1 \subset K$. So we can iterate again.

After countably many steps, we obtain a pair $(R_\infty, J_\infty) \subset (C, K)$ such that J_∞ does admit q-divided powers, which we denote ρ. In this way, $(R_\infty, J_\infty, \rho) \to (C, K, \kappa)$ is a map of q-divided power algebras. Hence, to verify condition (2) in Lemma 2.2 we just need to bound the cardinality of all such R_∞'s in terms of (B, J) and (A, I) (i.e., uniformly in (C, K)). But this is clear from the construction since R_0 is a quotient of B, and we iterated the finite steps in the construction of R_∞ at most countably many times. □

Definition 3.2 Let (A, I, η) be a q-divided power algebra. Let $A \to B$ be a $\mathbf{Z}[q]$-algebra map. Let $J \subset B$ be an ideal with $IB \subset J$. The q-divided power algebra (B', J', δ) over (A, I, η) constructed in Lemma 3.1 is called the *q-divided power envelope of J in B relative to* (A, I) and is denoted $(D_{B,q}(J), \bar{J})$.

We let (A, I, η) be a divided power ring. Let (B, J) be a pair over (A, I). For emphasis, we record the universal property of $D_{B,q}(J)$ explicitly now. For any morphism of pairs $f : (B, J) \to (C, K)$ over (A, I), with (C, K, κ) a q-divided power algebra over (A, I, η), there exists a unique morphism of q-divided power algebras:

$$(D_{B,q}(J), \bar{J}, \delta) \to (C, K, \kappa)$$

extending $(B, J) \to (C, K)$. By "extending" we mean that the composition of the morphisms of pairs

$$(B, J) \to (D_{B,q}(J), \bar{J}) \to (C, K)$$

is equal to f. In other words, $(B, J) \to (D_{B,q}(J), \bar{J})$ is the universal map from (B, J) to a q-divided power algebra over (A, I, η) (it corresponds to the identity on (B, J)).

As a special case (corresponding to $K = 0$), there is a unique B-algebra map

$$D_{B,q}(J) \to B/J$$

carrying $\bar{J}$ to 0, and it identifies $D_{B,q}(J)/\bar{J}$ with B/J (as is easily seen via the universal property). We shall often times omit to mention $\bar{J}$ and the q-divided power structure δ on $\bar{J}$.

Remark 3.3 The arguments around the countably infinite process in the proof of Lemma 3.1 imply that $D_{B,q}(J)$ is obtained by the following process. We begin with its subring B_0 given by the image of B, and the ideal $J_0 = JB_0 \subset \bar{J}$. Then we define $B_1 \subset D_{J,q}(B)$ to be generated over B_0 by the q-divided powers in $D_{J,q}(B)$, and J_1 to be the ideal in B_1 generated by the q-divided powers of the elements of $J_0B_1 \subset \bar{J}$. We keep going, countably many times, and in the (co)limit we reach a q-divided power sub-object $(B_\infty, J_\infty) \subset (D_{J,q}(B), \bar{J})$. By universal property considerations, $B_\infty = D_{J,q}(B)$ and $J_\infty = \bar{J}$.

Since a $\mathbf{Z}$-module M is flat if and only if it is n-torsion free for all $n \geq 1$, we introduce the following convenient terminology.

Definition 3.4 A $\mathbf{Z}[q]$-module is called *q-flat* if it is $[n]_q$-torsion-free for all $n \geq 1$.

For the multiplicative set $S \subset \mathbf{Z}[q] - \{0\}$ generated by the elements $[n]_q$ for $n \geq 1$ and any $\mathbf{Z}[q]$-module M, the image of $M \to S^{-1}M$ is a q-flat quotient of M and dominates all others. We denote it $M_{q\text{fl}}$. Here is the general behavior of q-divided power envelopes with respect to quotients.

Proposition 3.5 *Let (A, I, η) be a q-divided power ring. Let $f : B' \to B$ be a surjection of A-algebras, with kernel K. Let $J \subset B$ be an ideal containing IB, and $J' = f^{-1}(J) \subset B'$.*

Let K' be the smallest ideal in $D_{B',q}(J')$ satisfying $KD_{B',q'}(J') \subset K' \subset \bar{J}'$ and admitting q-divided powers. Then the natural map $D_{B',q}(J') \to D_{B,q}(J)$ is a surjection carrying $\bar{J}'$ onto $\bar{J}$ and killing K', with the resulting quotient map $D_{B',q}(J')/K' \to D_{B,q}(J)$ inducing an isomorphism between maximal q-flat quotients.

Proof. Write $E = D_{B,q}(J)$ and $E' = D_{B',q}(J')$. The universal property of E' gives us a homomorphism $E' \to E$ of q-divided power algebras. As $B' \to B$ and $J' \to J$ are surjective, we see that the natural map $\varphi : E' \to E$ is surjective because of Remark 3.3. Since K' is generated from the subideal $K'_0 := KD_{B',q}(J') = KE' \subset \bar{J}'$ by the evident countably infinite process as in Remark 3.3 (i.e., form the ideal K'_1 inside $\bar{J}'$ generated by q-divided powers of all elements of K'_0, and then the ideal K'_2 generated by all q-divided powers of elements of K'_1, and so on), clearly K' is in the kernel of φ. Hence, we obtain a natural homomorphism $E'/K' \to E$ that inherits surjectivity from that of $B' \to B$ (due to Remark 3.3).

There is a naturally induced surjection $(E'/K')_{q\mathrm{fl}} \to E_{q\mathrm{fl}}$ which we want to show is injective. Note also that E'/K' is a B-algebra since $B = B'/K$ and $KE' \subset K'$. Since in the maximal q-flat quotient $(E'/K')_{q\mathrm{fl}}$ the elements x' of the image $\bar{J}'_{q\mathrm{fl}}$ of $\bar{J}'$ satisfy $x'^n \in [n]_q!\bar{J}'_{q\mathrm{fl}}$, so the ideal $\bar{J}'_{q\mathrm{fl}}$ admits q-divided powers, by the universal property of E we also obtain a map of divided power algebras $E \to (E'/K')_{q\mathrm{fl}}$ over (A, I). The target is q-flat, so this uniquely factors through $E_{q\mathrm{fl}}$. Via universality considerations, $(E'/K')_{q\mathrm{fl}} \to E_{q\mathrm{fl}} \to (E'/K')_{q\mathrm{fl}}$ is the identity map. $\square$

We now discusse an important relation between q-divided power envelopes and classical divided power envelopes. To do this, we require a $\mathbf{Z}$-flatness property analogous to [**BhSch**, Def. 16.2(3)].

Lemma 3.6 *If P is a polynomial ring in finitely many variables over a $\mathbf{Z}[q]$-smooth ring A, and J an ideal in P such that P/J is $\mathbf{Z}[q]$-smooth. Then for $D = D_{P,q}(J)$, the quotient $D_{q\mathrm{fl}}/(q-1)$ is $\mathbf{Z}$-flat.*

The proof of this involves messy computations and will be given elsewhere. A consequence for our needs is that for the ideal $\bar{J} \subset D$, its image J' in $D_{q\mathrm{fl}}/(q-1)$ admits divided powers in the usual sense since every $x \in \bar{J}$ satisfies $x^n \in [n]_q!\bar{J}$ and hence its image $x' \in J'$ satisfies $x'^n \in n!J'$ (so J' admits divided powers because $D_{q\mathrm{fl}}/(q-1)$ is $\mathbf{Z}$-flat).

We are finally ready to establish the following:

Proposition 3.7 *Let (A, I, η) be a q-divided power ring with $\mathbf{Z}[q]$-smooth A and (B, J) a pair over (A, I) with B and B/J smooth over A. Let $(D_{B,q}(J), J_D, \delta_D)$ be the q-divided power envelope of (B, J).*

Letting $(\overline{B}, \overline{J}) = (B/(q-1)B, J/(q-1)J)$ we have a unique isomorphism of divided power algebras over $(\overline{A}, \overline{I}, \gamma)$:

$$D_{B,q}(J)_{q\mathrm{fl}}/(q-1) \simeq D_{\overline{B}}(\overline{J})$$

carrying the canonical ideal of $D_{B,q}(J)$ onto the canonical ideal of $D_{\overline{B}}(\overline{J})$ via the universal property of divided power envelopes.

Proof. Let $D = D_{B,q}(J)_{q\mathrm{fl}}$. The canonical map $D_{B,q}(J) \to D_{\bar{B}}(\bar{J})$ has target that is $\mathbf{Z}$-smooth, hence $\mathbf{Z}$-flat and so q-flat since it is a $\mathbf{Z}[q]$-algebra killed by $q-1$. It is a surjection due to the concrete description of $D_{\bar{B}}(\bar{J})$ (thanks to our smoothness hypotheses), and by $\mathbf{Z}$-flatness of its target must factor through the q-flat quotient D. In this way we obtain a surjection $D/(q-1) \to D_{\bar{B}}(\bar{J})$, and our goal is to show that is an isomorphism.

By Lemma 3.6, the quotient $\bar{B}$-algebra $D/(q-1)$ is $\mathbf{Z}$-flat and so admits a divided power structure on the image J' of the canonical ideal of D (since $x'^n \in n!J'$ for all $x' \in J'$, due to the fact that $x^n \in [n]_q!J_D$ for all $x \in J_D$). Hence, by universality there is a canonical map $D_{\bar{B}}(\bar{J}) \to D/(q-1)$. This is surjective due to the concrete description of D as a rising tower (see Remark 3.3), and the composite map

$$D_{\bar{B}}(\bar{J}) \to D/(q-1) \to D_{\bar{B}}(\bar{J})$$

is the identity by universality. Hence, by surjectivity of both maps in this composition it follows that both maps are isomorphisms. $\square$

The preceding result will enable us to ultimately compare q-crystalline and de Rham cohomology, re-constructing the latter as the specialization at $q = 1$ of the former.

4. Examples and concluding remarks

In this subsection we record a few technical results that will enable us to make contact with the existing de Rham and crystalline theories once we develop q-crystalline cohomology. This section is mostly about examples and tools to build new q-divided power algebras from old ones.

We fix a q-divided power ring (A, I, η).

First, we discuss some cases in which we may extend q-divided powers across $\mathbf{Z}[q]$-algebra maps. Namely, suppose B is an A-algebra and consider the pair (B, IB). We say that η extends to (B, IB) if there exists $\eta' : IB \to IB$ a q-divided power structure, such that for every $x \in I \subset IB$ we have $\eta'_n(x) = \eta_n(x)$ for every $n \geq 0$.

We have:

Lemma 4.1 *Let (A, I, η) be a q-divided power ring, and B an A-algebra. Then η extends to B when I is principal and B is $[n]_q$-torsion free for every $n \geq 1$.*

Proof. Let a be a generator of I, so IB consists of elements ab for $b \in B$. By the q-flatness of B, extending η is equivalent to the condition $x^n \in [n]_q!IB$ for all $x \in IB$. Since $x = ab$, we have $x^n = a^n b^n = [n]_q!\eta_n(a)b^n \in [n]_q!IB$ as desired. $\square$

We now have the following:

Lemma 4.2 *Let (A, I, η) be a q-divided power ring, and (B, J) an algebra over (A, I). Assume that:*

(i) *B is $[n]_q$-torsion free for every $n \geq 1$.*

(i) *$q - 1$ is regular in B.*

Then $q-1$ is regular on $D_{B,q}(J)_{\text{qfl}}$. If moreover $q - 1$ is regular on A, then

$$D_{B,q}(J)_{\text{qfl}} \otimes^{\mathbf{L}}_A A/(q-1) = D_{B,q}(J) \otimes_A A/(q-1).$$

Proof. For the first statement: since the formation of q-divided power envelopes commutes with localization at multiplicative sets (due to universality), we have

$$D_{B,q}(J)_{\text{qfl}} \subset B\left[\frac{1}{[n]_q}, n \geq 1\right]$$

by assumption (i). Since $q-1$ is regular in $B\left[\frac{1}{[n]_q}, n \geq 1\right]$ by assumption (ii), it is regular in $D_{B,q}(J)_{\text{qfl}}$ too.

Finally, if $q-1$ is regular in A then the homology of $D_{B,q}(J)_{q\mathrm{fl}} \otimes^{\mathbf{L}}_A A/(q-1)$ is concentrated in degrees -1 and 0. In degree -1 it is $\mathrm{Tor}_1^A(A/(q-1), D_{B,q}(J)_{q\mathrm{fl}})$, and in degree 0 it is $D_{B,q}(J) \otimes_A A/(q-1)$. We only need to check that

$$\mathrm{Tor}_1^A(A/(q-1), D_{B,q}(J)_{q\mathrm{fl}}) = 0,$$

which holds since we have already seen that $q-1$ is regular in $D_{B,q}(J)_{q\mathrm{fl}}$. □

In our intended applications, it will always be the case that $q-1$ is regular, and so Lemma 4.2 applies. This Lemma will be useful when computing q-crystalline cohomology using Čech-Alexander complexes.

Already in the case $q-1$ is regular in A, with (A, I, η) a q-divided power ring, we have that the functor $(\cdot) \otimes^{\mathbf{L}}_A A/(q-1)$ commutes with totalization of complexes. This is true because regularity of $q-1$ in A implies that $(\cdot) \otimes^{\mathbf{L}}_A A/(q-1)$ has finite homological dimension, and hence commutes with totalizing a simplicial A-module. Lemma 4.1 is useful to compare the $[p]_q$-completion of our q-crystalline theory, further base changed to Fontaine's A_{inf}, with the complex $A\Omega$ of [**BMS**]; this will be discussed elsewhere.

CHAPTER 3

Completions and Čech-Alexander complexes

In this chapter we collect a few results on q-divided power envelopes in order to use them to prove and use Poincaré Lemmas and relate Čech-Alexander complexes to q-de Rham complexes.

1. q-pd adic filtration on q-divided power algebras

Recall that given a q-pair (A, I), that is, a $\mathbf{Z}[q]$-algebra A and an ideal $I \subset A$, there exists a universal way to “enlarge" A enough to ensure I is contained in an ideal that carries q-divided powers. That is the q-divided power envelop $(D_{A,q}(J), \bar{J})$.

Definition 1.1 Let (A, I, η) be a q-divided power ring. We denote by $I^{\langle n\rangle}$ the ideal of A defined by the following three properties:

(i) $I^{\langle n\rangle}$ contains the ideal of A generated by all the elements of the form:

$$\eta_{i_1}(x_1) \cdots \eta_{i_k}(x_k)$$

where $x_1, \dots, x_k \in I$ and $i_1 + \cdots + i_k \geq n$.

(ii) $I^{\langle n\rangle}$ is contained in I and stable under the q-divided powers on I.

(iii) $I^{\langle n\rangle}$ is minimal with respect to the two properties above.

We set $I^{\langle 0\rangle} = A$ and we have $I^{\langle 1\rangle} = I$. The sequence of ideals $I^{\langle n\rangle} \subset A$ forms a decreasing filtration on A upon varying n. We call this the *q-pd adic filtration on A.*

Moreover, we say:

(a) (A, I, η) is *q-pd nilpotent* if there exists an integer $n \geq 0$ such that $I^{\langle n\rangle} = 0$.

(b) (A, I, η) is *q-pd quasi-nilpotent* if there exist an integer $n \geq 0$, and $g \in \mathbf{Z}[q]$, with $g(1) \neq 0$, such that:

$$g(q) \cdot I^{\langle n\rangle}$$

We call the linear topology on A generated by the filtration $F_\bullet A$ with $F_t A := [t]_q! I^{\langle t\rangle}$ for all $t \geq 1$, the *q-pd adic topology.*

Before moving on, we are going to make an observation that will prove useful.

Remark 1.2 Let (B, J) be an (A, I)-algebra. Let $x \in J$ and denote by x also its image in $\bar{J} \subset D_{B,q}(J)$. For all integers $m, n \geq 0$, $m \neq 0$, and for $N = m(n+1)$, x^N is contained in $[m]_q \cdot \bar{J}^{\langle n\rangle}$.

Indeed, $\eta_N(x)$ is an element of $\bar{J}^{\langle n\rangle}$, and $[N]_q! \cdot \eta_N(x)$ is an element of $[m]_q \cdot \bar{J}^{\langle n\rangle}$. However, we know $[N]_q! \cdot \eta_N(x) = x^N$.

In other words, the image of x^N in $D_{B,q}(J)/[m]_q\bar{J}^{\langle n\rangle}$ is zero. This remark, as simple as it is, should give an idea as to why we modeled our theory of q-divided powers around the operations "$x^n/[n]_q!$": we wish to retain the ability to recover powers of x by multiplying its q-divided powers by large enough q-factorials. The importance of this simple fact will become clear below.

We discuss a number of examples.

Example 1.3 Let (A, I, η) be a q-divided power ring. Assume that for every integer $n \geq 1$, $[n]_q \in A^\times$. Then $I^{\langle n\rangle} = I^n$ for every $n \geq 0$.

Example 1.4 Let (A, I, η) be a q-divided power ring such that $q = 1$ in A. Then, $I^{\langle n\rangle} = I^{[n]}$, and $g(q) \cdot I^{\langle n\rangle} = mI^{[n]}$ with $g(q) = g(1) = m$, and $I^{[n]}$ denoting the n-th divided power ideal, as in [**Berthelot**, §3.1]. In particular, the q-pd adic filtration on A is the classical pd-adic filtration of loc.cit., and our notion of q-pd nilpotence, resp. q-pd quasi-nilpotence, agree with pd-nilpotence and quasi pd-nilpotence (note the order of the terms) of [**Berthelot**, §3.1 Def. 3.1.1].

Example 1.5 Let (A, I, η) be a $\mathbf{Z}[q]$-flat q-divided power ring. Assume $q-1$ is regular in A. Then upon denoting $I^{[n]}$ the image of $I^{\langle n\rangle}/(q-1)I^{\langle n\rangle} \to A/(q-1)A$, and γ the divided power structure on I_0, the image of $I/(q-1)I \to A/(q-1)A$, we have that $I^{[n]}$ is generated by the elements:

$$\gamma_{i_1}(x_1) \cdots \gamma_{i_k}(x_k)$$

for $x_1, \ldots, x_k \in I_0$ and $i_1 + \cdots + i_k \geq n$. Generated, because of the additivity axiom for divided powers, whence the ideal of A contained in I_0 and generated by these elements already carries divided powers. In particular, $I_0^{[n]} = I^{[n]}$.

We now prove the following result, after which we shall have most of the tools to establish our main results.

Proposition 1.6 *Let A be a $\mathbf{Z}[q]$-algebra, B an A-algebra, $J \subset B$ an ideal and $C = B/J$. Assume:*

(1) *J is generated by a regular sequence $x_1, \ldots, x_d$.*

(2) *the homomorphisms $B/J^n \to C$ have A-linear sections for every $n \geq 1$.*

Then upon denoting by K the minimal q-divided power ideal containing the ideal generated by $t_1, \ldots, t_d$ in $D_{C[t_1,\ldots,t_d],q}((t_1, \ldots, t_d))$, for every integer $m, n \geq 0$, $m \neq 0$, we have:

$$(D_{C[t_1,\ldots,t_d],q}((t_1, \ldots, t_d))/[m]_q K^{\langle n\rangle})_{q\text{fl}} \simeq (D_{B,q}(J)/[m]_q \bar{J}^{\langle n\rangle})_{q\text{fl}}.$$

Proof. We recall that an ideal $J \subset B$ generated by a regular sequence $(x_1, \ldots, x_d)$ satisfies:

(a) J/J^2 is free of rank d as a B/J-module.

(b) the natural map of graded B-algebras:

$$\mathrm{Sym}_{B/J}(J/J^2) \to \mathrm{gr}_J(B)$$

is an isomorphism.

In particular, J^t/J^{t+1} is free as a B/J-module for every integer $t \geq 1$.

Let N be $m(n+1)$. By Remark 1.2, the image of x^N in $D_{B,q}(J)/[m]_q K^{\langle n\rangle}$ is zero. Since J is generated by d elements, the image of J^{Nd} in $D_{B,q}(J)/[m]_q K^{\langle n\rangle}$ is zero too. By Proposition 3.5, we have a canonical isomorphism:

$$(D_{B/J^{Nd},q}(J/J^{Nd})/[m]_q(J/J^{Nd})^{\langle n\rangle}_{\mathrm{qfl}} \xrightarrow{\simeq} (D_{B,q}(J)/[m]_q J^{\langle n\rangle})_{\mathrm{qfl}}.$$

By assumption (2), we may choose a section of $B/J^{Nd} \to C$ and give B/J^{Nd} a C-algebra structure, so as to define a C-algebra map:

$$C[t_1,\ldots,t_d]/I^{Nd} \to B/J^{Nd}$$

sending t_i to x_i for all $1 \leq i \leq d$. This is an isomorphism by assumption (1). We therefore obtain an isomorphism:

$$D_{C[t_1,\ldots,t_d]/I^{Nd},q}(I/I^{Nd}) \xrightarrow{\simeq} D_{B/J^{Nd},q}(J/J^{Nd})$$

and another application of Proposition 3.5 gives the contention. □

Remark 1.7 We note that, if both B and C are $\mathbf{Z}[q]$-flat in Proposition 1.6, then so is B/J^n for every $n \geq 0$. Indeed, one proceeds by induction on n using the short exact sequences of B/J^n-modules:

$$0 \to J^n/J^{n+1} \to B/J^{n+1} \to B/J^n \to 0$$

using that J^n/J^{n+1} is free of finite type as a B/J-module, hence $\mathbf{Z}[q]$-flat. By Lemma 4.2, $(D_{B/J^n,q}(J/J^n))_{\mathrm{qfl}}$ is $(q-1)$-torsion free.

Now assume B and C is $\mathbf{Z}[q]$-smooth and B/J is $\mathbf{Z}[q]$-smooth. In this case, by Lemma 3.6, the quotient $(D_{C[t_1,\ldots,t_d],q}((t_1,\ldots,t_d)))_{\mathrm{qfl}}/(q-1)$ is the divided power polynomial algebra $(C/(q-1)C)\langle \bar{t}_1,\ldots,\bar{t}_d\rangle$, and the ideal $[m]_q K^{\langle n\rangle}$ specializes to the ideal of $(C/(q-1)C)\langle \bar{t}_1,\ldots,\bar{t}_d\rangle$ given by $m(K/(q-1)K)^{[n]}$. Moreover,

$$(D_{B,q}(J)/[m]_q\bar{J}^{\langle n\rangle})_{\mathrm{qfl}}/(q-1) \simeq D_{B/(q-1)B}(J/(q-1)J)/m\overline{J/(q-1)J)}^{[n]}.$$

By regularity of $q-1$, the isomorphism in Proposition 1.6 yields an isomorphism:

$$(C/(q-1)C)\langle \bar{t}_1,\ldots,\bar{t}_d\rangle/m(K/(q-1)K)^{[n]} \xrightarrow{\simeq} D_{B/(q-1)B}(J/(q-1)J)/mJ/(q\bar{-}1)J)^{[n]}.$$

This isomorphism is [**Berthelot**, Prop. 3.5.1,(i)].

2. Čech-Alexander complexes

In this section we study two Čech-Alexander complexes, that later we will interpret as the cohomology of a structure sheaf on an appropriate q-crystalline site.

Lemma 2.1 *Let (A, I, η) be a q-divided power ring. Assume A is $\mathbf{Z}[q]$-flat. Then for every $n \geq 0$, η_n is continuous for the $(q-1)$-adic topology on I.*

Proof. Recall that by Lemma 1.14, carrying q-divided powers is equivalent, for I, to the property: for every $x \in I$, $x^n \in [n]_q!I$, for every integer $n \geq 1$. This crucially uses $[n]_q$-torsion freeness for every $n \geq 1$ and n-torsion freeness of $A/(q-1)A$ for every $n \geq 1$, which in our case follows by flatness over $\mathbf{Z}[q]$. However, $\mathbf{Z}[q]$-flatness of A implies $\mathbf{Z}[q]/(q-1)^t$-flatness of $A_t := A/(q-1)^t A$ for every $t \geq 0$. It follows that each A_t is also $[n]_q$-torsion free for every $n \geq 1$ and every $t \geq 0$. Indeed, q acts as $(\zeta_d)_{1 \neq d|n}$ on $\mathbf{Z}[q]/[n]_q \subset \prod_{1 \leq d|n} \mathbf{Z}[\zeta_d]$, which is therefore $(q-1)$-torsion free, whence $\mathrm{Tor}_1^{\mathbf{Z}[q]}(\mathbf{Z}[q]/[n]_q, \mathbf{Z}[q]/(q-1)^t) = 0$ for every $t \geq 0$. It follows multiplication by $[n]_q$ is injective on $\mathbf{Z}[q]/(q-1)^t$, and hence so is on A_t by flatness.

This implies that, upon calling $I_t := I/(q-1)^t I \subset A_t$, the maps $\eta_n : I \to I$ descend to $\eta_{n,t} : I_t \to I_t$ to make the following diagram of maps of sets commute:

$$\begin{tikzcd} I \arrow[r] \arrow[d, "\eta_n"'] & I_t \arrow[d, "\eta_{n,t}"] \\ I \arrow[r] & I_t \end{tikzcd}$$

One may check that the $\eta_{n,t}$'s are a q-divided power structure, for instance by Lemma 1.14. Let $x + (q-1)^t y \in I$. Then, modulo $(q-1)^t$, $\eta_n(x + (q-1)^t y)$ must agree with $\eta_{n,t}(x)$ by the existence of the maps $\eta_{n,t}$ and commutativity of the above diagram: the former is ensured by $[n]_q$-torsion freeness for every $n \geq 1$, and the latter by design of the $\eta_{n,t}$. It follows that $\eta_n(x+(q-1)^t y) - \eta_n(x) \in (q-1)^t A \cap I = (q-1)^t I$, for any $y \in I$, whence η_n is continuous in the $(q-1)$-adic topology on I. □

We need one more lemma along similar lines.

Lemma 2.2 *Let (A, I, η) be a q-divided power ring. Assume A is:*

(a) *$[n]_q$-torsion free for every $n \geq 1$.*

(b) *$(q-1)$-torsion free.*

Let P be an A-flat A-algebra and I an ideal of P. If $(q-1)\widetilde{P}$ is closed in the I-adic topology on $\widetilde{P}$, then, upon calling $D := D_{P,q}(I)_{q\mathrm{fl}}$, $(q-1)D$ is closed in the q-pd adic topology on D.

Proof. We observe that the assumptions on A ensure that $D \subset \widetilde{P}$, where $\widetilde{A} := A[1/[n]_q, n \geq 1]$ and $\widetilde{P} := P \otimes_A \widetilde{A}$. We write $\widetilde{P} := \widetilde{A}[S]$. Since $(q-1)\widetilde{P}$ is closed in the $I\widetilde{P}$-adic topology, we conclude by observing that for every integer $n \geq 0$, $I^{\langle n \rangle} \subseteq I\widetilde{P} \cap D$, the intersection taking place in $\widetilde{P}$. □

Lemma 2.3 *Let (A, I) be a q-pair. Assume that:*

(a) *both A and A/I are $\mathbf{Z}[q]$-flat.*

(b) *I^t/I^{t+1} is a flat A/I-module for every $t \geq 1$.*

Then any element of $\{q-1\} \cup \{[n]_q, n \geq 1\}$ is regular in $D_{A,q}(I)_{q\mathrm{fl}}/\bar{I}^{\langle t \rangle}$ for every $n \geq 0$.

Proof. Upon writing $\widetilde{A} := A[1/[n]_q, n \geq 1]$ and

$$\widetilde{A/I} := (A/I^t)[1/[n]_q, n \geq 1],$$

we have $D_{A,q}(I)_{q\text{fl}} \subset \widetilde{A}$, $\bar{I}^{\langle t \rangle} \subset I\widetilde{A}$, and $D_{A/I^t,q}(I/I^t)_{q\text{fl}} \simeq D_{A,q}(I)_{q\text{fl}}/I^{\langle t \rangle} \subset \widetilde{A/I^t}$, where we have used Lemma 3.5 and assumptions (a) and (b), this latter to argue that A/I^t is $\mathbf{Z}[q]$-flat for all $t \geq 0$. Since $q-1$ is regular in each $\widetilde{A/I^t}$, we conclude. □

Remark 2.4 We record here a consequence of the above preparations. Suppose P is $\mathbf{Z}[q]$-flat and $J \subset P$ is an ideal. Denote by $\widetilde{P}$ the localization $P[1/[n]_q, n \geq 1]$ and W the $J\widetilde{P}$-adic completion of $\widetilde{P}$. Assume that:

(a) P/J is $\mathbf{Z}[q]$-flat.

(b) J^t/J^{t+1} is a flat P/J-module for every $t \geq 1$.

(c) $(q-1)W$ is closed in the JW-adic topology.

We denote by $\widehat{D}_{P,q}(J)$ the completion of $D_{P,q}(J)$ with respect to the q-pd adic topology. We have

$$\widehat{D}_{P,q}(J) = \varprojlim_{t \geq 1} D_{P,q}(J)_{q\text{fl}}/[t]_q! \bar{J}^{\langle t \rangle}. \tag{2.0.1}$$

The element $q-1$ is regular in $\widehat{D}_{P,q}(J)$ too. From $\mathbf{Z}[q]$-flatness of P/J^t for every $t \geq 1$, we deduce that $\widehat{D}_{P,q}(J)$ is a subalgebra of the $I\widetilde{P}$-adic completion of $\widetilde{P}$. Then by the proof of Lemma 2.2, $(q-1)\widehat{D}_{P,q}(J)$ is a closed ideal.

We deduce:

$$\widehat{D}_{P,q}(J)/(q-1) \simeq \varprojlim_{t \geq 1} D_{P/(q-1)P}(J/(q-1)P)/t! \overline{(J/(q-1)J)}^{\langle t \rangle}.$$

We denote the right side of the above isomorphism by $\widehat{D}_{P/(q-1)P}(J/(q-1)J)$.

We introduce our first Čech-Alexander complex.

Construction 2.5 Let (A, I, η) be a q-divided power ring, and R a smooth A-algebra. Choose an A-algebra P that is smooth over A, and an A-algebra surjection $P \to R$. Let $P^\bullet$ be the Čech nerve of $A \to P$. The kernel $J(n)$ of each augmentation $P^{\otimes_A (n+1)} \to P \to R$ is, locally on Spec(P), generated by a regular sequence relative to A. We obtain a cosimplicial diagram

$$(D_{P^\bullet,q}(J(\bullet))_{q\text{fl}}, \bar{J}(\bullet))$$

in the category of q-divided power algebras over (A, I, η).

Since the co-degeneracies are maps of q-divided power algebras, the differentials in the associated complex

$$C_{q\text{-cris}}(R/A, P^\bullet)$$

of A-modules are continuous with respect to the q-pd adic topologies of each $D_{P^\bullet,q}(J(\bullet))_{q\text{fl}}$. We denote by

$$\widehat{C}_{q\text{-cris}}(R/A, P^\bullet)$$

the resulting term-wise completion with respect to the q-pd adic topology. We call the former, the *Čech-Alexander complex of R over A*, and the latter, the *completed Čech-Alexander complex of R over A*. We further denote $C_q(R/A, P^\bullet)$ the $(q-1)$-adic completion of $\widehat{C}_{q\text{-cris}}(R/A, P^\bullet)$.

Remark 2.6 We assume that (A, I, η) a q-divided power ring, such that $(q-1)A$ is closed in the q-pd adic topology on A, and A is $\mathbf{Z}[q]$-flat. We let P be a flat A-algebra. We further assume that we are in the situation where each $(P^\bullet, J(\bullet))$ satisfies the assumptions in Remark 2.4, so then

$$\widehat{C}_{q\text{-cris}}(R/A, P^\bullet) \otimes^{\mathbf{L}}_{A^\wedge} A^\wedge/(q-1)A^\wedge = \widehat{C}_{q\text{-cris}}(R/A, P^\bullet) \otimes_{A^\wedge} A^\wedge/(q-1)A^\wedge$$

and the right side admits an isomorphism of complexes of A-modules:

$$\widehat{C}_{q\text{-cris}}(R/A, P^\bullet) \otimes_{A^\wedge} A^\wedge/(q-1)A^\wedge \simeq C_q(R/A, P^\bullet) \otimes_A A/(q-1).$$

By (2.0.1), $\widehat{C}_{q\text{-cris}}(R/A, P^\bullet) \otimes_{A^\wedge} A^\wedge/(q-1)A^\wedge$ is the complex of $A^\wedge/(q-1)A^\wedge$-modules given by the completion of the cosimplicial divided power ring

$$(D_{P^\bullet/(q-1)P^\bullet}(J(\bullet)/(q-1)J(\bullet)), \bar{J}(\bullet)/(q-1)\bar{J}(\bullet))$$

with respect to the pd-adic topology on each term.

The case of main interest to us will be $A = A_0[q]$ for a smooth $\mathbf{Z}[1/N]$-algebra A_0, with the ideal $I = (0)$ and the trivial q-divided power structure. If R_0 is a smooth A_0-algebra, R will usually be $R_0[q]$, and P will be of the form $P_0[q]$ with P_0 a smooth A_0-algebra equipped with étale coordinates:

$$g_0 : A_0[x_1, \ldots, x_d] \to P_0$$

and an A_0-algebra surjection $P_0 \to R_0$. In particular, A, P and R are Noetherian, and P is (or can be arranged to be) an integral domain. In particular, for any ideal J of P, the J-adic completion of P is P-flat and the $(J/(q-1)J)$-adic completion of P_0 is $(J/(q-1)J)P_0$-adically separated (so $(q-1)P$ is closed for the J-adic topology).

We will often replace A, P and R by their $(q-1)$-adic completions.

Ultimately, our goal is to show that for certain A-algebras R, the complex

$$C_q(R/A, P^\bullet)$$

is quasi-isomorphic to a q-de Rham complex (in the sense of Definition 2.8 below) as a complex of $A^\wedge$-modules, $A^\wedge$ being the completion of A with respect to an appropriate linear topology coarser than the $(q-1)$-adic.

We now recall a construction from [**BhSch**, §16] that is a slight generalization of the original construction of the q-de Rham complex in [**Aom**].

Construction 2.7 A *framed* A-algebra is a pair (P, S), where P is an smooth A-algebra and $S \subset P$ is a finite set, and P comes with a morphism:

$$g : A[S] := A[\{t_s\}_{s\in S}] \to P$$

that is étale.

For each $s \in S$, there is a unique A-algebra automorphism γ_s of $A[S]$ defined by scaling $t_s \mapsto qt_s$ and leaving $t_{s'}$ fixed for $s' \neq s$. We now assume A is $(q-1)$-adically complete and separated.

As the automorphism γ_s is congruent to the identity modulo the topologically nilpotent element $q-1$, it extends uniquely to a A-algebra automorphism γ_s of the $(q-1)$-adic completion $P^\wedge$ of P that is also congruent to the identity modulo $q-1$.

We now assume A is $\mathbf{Z}[q]$-flat, whence so is P, and hence $q-1$ is not a zero divisor in P. It therefore is regular in in $P^\wedge$ too. Moreover, the étaleness of g yields that t_s is not a zero-divisor in $P/(q-1)^t P$ for all $t \geq 1$, so t_s is not a zero-divisor in $P^\wedge$. In particular, $qt_s - t_s = (q-1)t_s$ is a nonzerodivisor in $P^\wedge$, so we obtain a q-derivation $\nabla_{q,s} : P^\wedge \to P^\wedge$ given by the formula

$$\nabla_{q,s}(f) := \frac{\gamma_s(f) - f}{qt_s - t_s}.$$

Note that $\nabla_{q,s}$ lifts $\frac{\partial}{\partial t_s}(\cdot)$ on $A[S]/(q-1)A[S]$ and hence on $P/(q-1)P$: this is clear for $A[S]$ by explicit computation and follows for P by étaleness. Likewise, the $\nabla_{q,s}$'s commute with each other since this reduces to the case of $A[S]$.

We can assemble these q-derivations into a A-linear map

$$\nabla_q : P^\wedge \to \widehat{\Omega}^1_{P^\wedge/A} := \widehat{\bigoplus}_{s\in S} P^\wedge dt_s$$

given by the rule $\nabla_q(f) := \sum \nabla_{q,s}(f) dt_s$. Taking the completed Koszul complex on the commuting endomorphisms $\nabla_{q,s}$ (which are q-derivatives, so their non-linearity does not obstruct carrying out the Koszul construction).

Definition 2.8 We define the *q-de Rham complex*

$$q\Omega^{*,\square}_{P/A} := \left(P^\wedge \xrightarrow{\nabla_q} \widehat{\Omega}^1_{P^\wedge/A} \xrightarrow{\nabla_q} \widehat{\Omega}^2_{P^\wedge/A} \to \ldots\right),$$

regarded as a chain complex of A-modules.

The q-de Rham complex has the following functoriality property: given two framed A-algebras (P,S) and (P',S') as well as a map $P \to P'$ of A-algebras carrying S into S', we obtain an induced morphism $q\Omega^{*,\square}_{P/A} \to q\Omega^{*,\square}_{P'/A}$ of chain complexes.

We now wish to extend Definition 2.8 to the case of q-divided power algebras. To this end, we let (A, I, η) be a q-divided power ring, and we assume A is $\mathbf{Z}[q]$-flat, so that Lemma 2.3 applies and η extends to the $(q-1)$-adic completion of A. That such extension is again a q-divided power structure can be shown directly (we omit this verification).

Construction 2.9 [The q-de Rham complex of a framed q-divided power algebra] A *framed q-divided power algebra* is a triple (P,S,J), where (P,S) is a framed A-algebra as in Construction 2.7 and $J \subset P$ is an ideal. Let $D_{P,q}(A)$ be the q-divided power envelope. We now require the following:

Lemma 2.10 *For each $s \in S$, assume the automorphism γ_s of $P^\wedge$ preserves $J^\wedge$. Then γ_s extends uniquely to an automorphism of $D_{P^\wedge,q}(J^\wedge)_{q\mathrm{fl}}$ that is congruent to the identity modulo $qt_s - t_s$. In particular, it extends to the $(q-1)$-adic completion of $D_{P^\wedge,q}(J^\wedge)_{q\mathrm{fl}}$ as an automorphism that is congruent to the identity modulo $qt_s - t_s$.*

Proof. This is a direct consequence of the universal property of q-divided power envelopes. Moreover, by $\mathbf{Z}[q]$-flatness of A, and hence of P, we have that $q-1$ is regular in $D_{P^\wedge,q}(J^\wedge)$ and hence we still have:

$$D_{P^\wedge,q}(J^\wedge)_{q\mathrm{fl}} \otimes_A A/(q-1)A \simeq D_{P/(q-1)P}(J/(q-1)J)$$

in which of course the extension of γ_s reduces to the identity. $\square$

We can thus define q-derivatives $\nabla_{q,s}$ of $D_{P^\wedge,q}(J^\wedge)_{\mathrm{qfl}}$ by the same formula as in Construction 2.7. These derivatives lift $\frac{\partial}{\partial t_s}$ on $D_{P^\wedge,q}(J^\wedge)/(q-1)$: this follows by identifying the latter as the divided power envelope of $(P/(q-1)P, J/(q-1)J)$ with $P/(q-1)P$ étale over the $\mathbf{Z}$-flat $(A/(q-1)A)[S]$. Taking the Koszul complex of the commuting endomorphisms $\nabla_{q,s}$ as in Construction 2.7, we obtain:

Definition 2.11 We define a *q-dR complex* $q\Omega^{*,\square}_{D_{P^\wedge,q}(J^\wedge)/A}$ to be

$$D_{P^\wedge,q}(J^\wedge)^\wedge_{\mathrm{qfl}} \xrightarrow{\nabla_q} D_{P^\wedge,q}(J^\wedge)^\wedge_{\mathrm{qfl}}\widehat{\otimes}_{P^\wedge}\widehat{\Omega}^1_{P^\wedge/A} \xrightarrow{\nabla_q} D_{P^\wedge,q}(J^\wedge)^\wedge_{\mathrm{qfl}}\widehat{\otimes}_{P^\wedge}\widehat{\Omega}^2_{P^\wedge/A} \to \dots,$$

regarded as a chain complex of A-modules.

The construction $(P,S,J) \mapsto q\Omega^{*,\square}_{D_{P^\wedge,q}(J^\wedge)/A}$ has the following functoriality property: given two framed q-divided power algebras (P,S,J) and (P',S',J') and a map $P \to P'$ of A-algebras carrying S into S' and J into J', we obtain an induced morphism $q\Omega^{*,\square}_{D_{P^\wedge,q}(J^\wedge)/A} \to q\Omega^{*,\square}_{D_{P',\wedge,q}(J',\wedge)/D}$.

If the $(q-1)$-adic completion of J is stable in the $(q-1)$-adic completion $P^\wedge$ of P under γ_s for every $s \in S$, then we define $q\widehat{\Omega}^{*,\square}_{D_q(J_D)/A}$ to be the completion of $q\Omega^{*,\square}_{D_{P^\wedge,q}(J^\wedge)/A}$ with respect to the q-pd adic topology on each term. The stability assumption on $J^\wedge$ under the γ_s, together with the design of the differentials, implies the differentials are continuous with respect to the q-pd adic topology, and hence they extend to the completion.

Remark 2.12 We note that by Lemma 2.6, when A is smooth over $\mathbf{Z}[q]$, P is A-smooth equipped with étale coordinates, and J is an ideal satisfying the above stability assumption then we have:

$$q\widehat{\Omega}^{*,\square}_{D_q(J_D)/A} \otimes_A A/(q-1)A \simeq \widehat{\Omega}^*_{\widehat{D}_{P/(q-1)P}(J/(q-1)J)/(A/(q-1)A)},$$

the completion of the divided power de Rham complex $\Omega^*_{\widehat{D}_{P/(q-1)P}(J/(q-1)J)/(A/(q-1)A)}$ with respect to the pd-adic topology.

3. q-crystalline site

We now construct the q-crystalline site for an affine scheme. The basic situation in which we discuss this material involves *q-divided power thickenings*: these are q-divided power algebras (A,I,η) for which $[n]_q!I^{\langle n\rangle} = 0$ for large enough $n \geq 1$, and A and I are $(q-1)$-adically separated and complete.

Definition 3.1 We let (A,I,η) be a q-divided power ring. We say (A,I,η) is $(q-1)$-*adically complete and separated* if A and I are complete complete and separated in the $(q-1)$-adic topology on A and I respectively.

(1) For a cardinal κ, we say a q-divided power algebra (B,J,δ) over (A,I,η) is *κ-small* if $|B| \leq \kappa$.

(2) We denote by $q\mathrm{CRIS}(A)$ the category whose objects are κ-small and $(q-1)$-adically complete and separated q-divided power algebras (B,J,δ) over (A,I,η), such that $[n]_q!J^{\langle n\rangle}B = 0$ for some $n \geq 1$, and whose morphisms are morphisms of q-divided power algebras.

We equip $q\mathrm{CRIS}(A)$ with the Grothendieck topology generated by formal affine Zariski coverings:

$$\{\mathrm{Spf}(B_i) \to \mathrm{Spf}(B)\}_{i \in \Sigma}$$

for Σ a finite index set.

(3) For a $(q-1)$-adically complete and separated A-algebra R, we denote by $q\mathrm{Cris}(R/A)$ the full subcategory consisting of (B, J, δ) in $q\mathrm{CRIS}(A)$ such that $\mathrm{Spf}(B/J)$ is a formal affine open of $\mathrm{Spf}(R)$ as $(q-1)$-adic formal schemes over A. We endow $q\mathrm{Cris}(R/A)$ with the Grothendieck topology induced from that of $q\mathrm{CRIS}(A)$.

We call the sites $q\mathrm{CRIS}(A)$ (resp. $q\mathrm{Cris}(R/A)$) the big (resp. small) q-crystalline site of A (resp. R over A). We also denote the small q-crystalline site of R over A by $(R/A)_{q\text{-cris}}$.

The small q-crystalline site $(R/A)_{q\text{-cris}}$ carries a ringed structure $\mathscr{O}_{q\text{-cris}}$ defined as the sheaf of topological A-algebras on $(R/A)_{q\text{-cris}}$ determined by the assignment: $(B, J, \delta) \mapsto B$. The *q-crystalline cohomology of R relative to A*, denoted $q\Omega_{R/A}$, is defined as

$$q\Omega_{R/A} := \mathbf{R}\Gamma((R/A)_{q\text{-cris}}, \mathscr{O}_{q\text{-cris}}).$$

We have the following:

Proposition 3.2 *Let (A, I, η) be a q-divided power ring with $I = 0$ such that A is $(q-1)$-adically complete and separated as well as formally smooth over $\mathbf{Z}[\![q-1]\!]$ for the $(q-1)$-adic topology.*

Let P be a topologically finitely presented formally smooth A-algebra equipped with formally étale coordinates:

$$g : A[t_1, \ldots, t_d]^{\wedge,(q-1)} \to P.$$

Then we have:

$$q\Omega_{P/A} \simeq C_q(P/A, P^\bullet).$$

Proof. Note that representable functors are sheaves on $(P/A)_{q\text{-cris}}$. We shall freely identify objects of the underlying category of the site $(P/A)_{q\text{-cris}}$ with their representable sheaf on $(P/A)_{q\text{-cris}}$. We shall omit mentioning the q-divided power structures, leaving them understood from the context.

The map

$$\varinjlim_{n \gg 0} (\mathrm{Spec}\ D_{P \otimes_A P, q}(J(1))_{q\mathrm{fl}}/[n]_q!\bar{J}(1)^{\langle n \rangle}, \bar{J}(1)/[n]_q!\bar{J}(1)^{\langle n \rangle}) \to *$$

is an effective epimorphism of sheaves of sets. To see this, one needs to check that any object of the topos of $(P/A)_{q\text{-cris}}$ that covers the final object also receives an epimorphism from

$$(\mathrm{Spec}\ D_{P \otimes_A P, q}(J(1))_{q\mathrm{fl}}/[n]_q!\bar{J}(1)^{\langle n \rangle}, [n]_q!\bar{J}(1)/\bar{J}(1)^{\langle n \rangle})$$

for some n large enough. This is ultimately consequence of the universal property of q-divided power envelopes, a verification which we omit here.

Since filtered colimits and the Yoneda embedding commute with finite products, we have that the $(n+1)$-fold self product of the sheaf

$$\varinjlim_{t \gg 0} (\mathrm{Spec}\ D_{P \otimes_A P, q}(J(1))_{q\mathrm{fl}}/[t]_q!\bar{J}(1)^{\langle t \rangle}, \bar{J}(1)/[t]_q!\bar{J}(1)^{\langle t \rangle})$$

is isomorphic to

$$\varinjlim_{t\gg 0}(\mathrm{Spec}\ D_{P(n),q}(J(n))_{q\text{fl}}/[t]_q!\bar{J}(n)^{\langle t\rangle}, [t]_q!\bar{J}(n)/\bar{J}(n)^{\langle t\rangle}).$$

General topos theory implies that $q\Omega_{P/A}$ is quasi-isomorphic to the total complex associated to:

$$R\Gamma(\varinjlim_{t\gg 0}(\mathrm{Spec}\ D_{P(0),q}(J(0))_{q\text{fl}}/[t]_q!\bar{J}(0)^{\langle t\rangle}, \bar{J}(0)/[t]_q!\bar{J}(0)^{\langle t\rangle}), \mathscr{O}_{q\text{-cris}}) \to$$
$$\to R\Gamma(\varinjlim_{t\gg 0}(\mathrm{Spec}\ D_{P(1)}(J(1))_{q\text{fl}}/[t]_q!\bar{J}(1)^{\langle t\rangle}, \bar{J}(1)_{q\text{fl}}/[t]_q!\bar{J}(1)^{\langle t\rangle}, \mathscr{O}_{q\text{-cris}}) \to \cdots$$

The diagram of global sections of the sheaf $\mathscr{O}_{q\text{-cris}}$ evaluated on each member of the colimits in the foregoing satisfy the Mittag-Leffler condition because the resulting transition maps are surjective: this is due to the fact that the morphisms

$$(\mathrm{Spec}\ D_{P(\bullet),q}(J(\bullet))_{q\text{fl}}/[t]_q!\bar{J}(\bullet)^{\langle t\rangle}, \bar{J}(0)/[t]_q!\bar{J}(\bullet)^{\langle t\rangle}) \to$$
$$\to (\mathrm{Spec}\ D_{P(\bullet),q}(J(\bullet))_{q\text{fl}}/[t+1]_q!\bar{J}(\bullet)^{\langle t+1\rangle}, \bar{J}(\bullet)/[t+1]_q!\bar{J}(\bullet)^{\langle t+1\rangle})$$

are monomorphisms of sheaves of sets. Indeed, in this case the global sections of the structure sheaf evaluated on each colimit are isomorphic to the limit of the global sections of the structure sheaf evaluated on each Spec $D_{P(\bullet),q}(J(\bullet))_{q\text{fl}}/[t]_q!\bar{J}(\bullet)^{\langle t\rangle}$, and this limit has vanishing higher cohomology due to the Mittag-Leffler property.

Moreover, the higher cohomology of $\mathscr{O}_{q\text{-cris}}$ on each Spec $D_{P(\bullet),q}(J(\bullet))_{q\text{fl}}/[t]_q!\bar{J}(\bullet)^{\langle t\rangle}$ vanishes, as follows from vanishing of quasi-coherent cohomology on affine schemes, for which an analogous argument to that in [**Berthelot**] applies.

We may therefore identify the above total complex computing $q\Omega_{P/A}$ with the total complex of the bicomplex defined by $C_q(P/A, P^\bullet)$. ☐

Before proving Theorem 4.1, we need an aside concerning crystalline cohomology over arbitrary bases. We record here results that extend the crystalline formalism to the case of arbitrary base schemes: this is [**Berthelot**, Appendice], where it is sketched how to compare algebraic de Rham cohomology to crystalline cohomology when the base scheme is allowed to be arbitrary (not necessarily torsion).

Ultimately, the key point is [**Berthelot**, Prop. 3.5.1, (i)], which is our Proposition 1.6 with $q = 1$. Indeed, the result enables one to argue that the following map of complexes is a quasi-isomorphism:

$$D \to \left(D(n) \to \Omega^1_{D(n)/D} \to \Omega^2_{D(n)/D} \to \cdots\right)$$

where $D := \varprojlim_{t\geq 1} D_P(J)/t_!J^{[t]}$ and $D(n) := \varprojlim_{t\geq 1} D_{P(n)}(J(n))/t!J(n)^{[t]}$, with $P^\bullet \to R$ being the Čech nerve of a surjection $P \to R$ of smooth A-algebras, with P a polynomial ring over A.

The whole point behind [**Berthelot**, Prop. 3.5.1, (i)] is that every $D(n)$ is a certain formal power series algebra over D for every $n \geq 1$ (this description is the content of [**Berthelot**, Prop. 3.5.1, (i)] applied with B there taken to be our P and C there taken to be our R). That the Poincaré Lemma holds true for all the $D(n)$ over D is not obvious and requires an argument. One observes that the elements of $D(n)$ are formal power series in the free variables generating $D_{P(n)}(J(n))/t!J(n)^{[t]}$ over $D_P(J)/t!J^{[t]}$ such that for an arbitrary integer $N \geq 1$, all the coefficients of the high order terms eventually all become divisible by N. The same argument for the Poincaré Lemma for formal power series algebras containing $\mathbf{Q}$ now carries over upon observing that the type of power series described above is stable under "integration".

With the above in mind, it is a simple task to show that cohomology of the structure sheaf on the crystalline site defined as in [**Berthelot**, Appendice], using quasi pd-nilpotent thickenings as opposed to ordinary thickenings, is again computed by a Čech-Alexander complex. Moreover, via the Poincaré Lemma (proved upon bearing in mind the changes above to the usual argument), one can still prove that cohomology of the de Rham complex computes crystalline cohomology of the crystalline structure sheaf.

4. Main result

Here finally is main theoretical result, from which we will then draw some interesting conclusions (going beyond the affine case).

Theorem 4.1 *For A a smooth $\mathbf{Z}[q]$-algebra and $A^\wedge$ its $(q-1)$-adic completion, P a smooth A-algebra equipped with étale coordinates $g : A[t_1, \ldots, t_d] \to P$, there exists a canonical quasi-isomorphism of complexes of $A^\wedge$-modules:*

$$C_q(P/A, P^\bullet) \simeq q\widehat{\Omega}^{*,\square}_{P^\wedge/A^\wedge}$$

The reason why this result is of interest is that it provides a site-theoretic interpretation for the complex $C_q(P/A, P^\bullet)$ as the derived global sections of a structure sheaf on an appropriate q-crystalline site.

Proof. As in the proof of Proposition 3.2, we denote by

$$P(n) := (P \otimes_A \cdots \otimes_A P)^\wedge$$

the $(n+1)$-fold tensor product of P over A, and define $J(n)$ to be with kernel of its augmentation to $P^\wedge$. Note that all these ideals are stable under the automorphisms γ_s coming from the étale coordinates on every $P(n)$.

We therefore write $T^{\bullet,*}$ the cosimplicial complex of A-modules:

$$T^{\bullet,*} := q\widehat{\Omega}^{*,\square}_{D_{J^\bullet}(P^\bullet)/A}$$

The cosimplicial A-module $T^{\bullet,i}$ is quasi-isomorphic to the zero complex for all $i > 0$, by derived Nakayama, since it is derived $(q-1)$-adically complete and $T^{\bullet,i}/(q-1)$ is indeed homotopic to zero as a complex of $A/(q-1)A$-modules. This latter assertion follows from the fact that each $P(n)/(q-1)$ is a smooth $A/(q-1)A$-algebra with étale coordinates, and the argument in [**BdJ**, Example 2.16] (or [**BhCdR**, Lemma 2.5]) adapts to this case. This implies that the total complex associated to $T^{\bullet,*}$ is quasi-isomorphic to $C^\bullet_q(P/A)$.

On the other hand the divided power Poincaré Lemma applies to $q\widehat{\Omega}^{*,\square}_{D_{J^\bullet}(P^\bullet)/A}/(q-1)$ and hence again by derived Nakayama the cosimplicial transition maps $T^{i,*} \to T^{j,*}$ are quasi-isomorphisms. The total complex associated to $T^{\bullet,*}$ is therefore quasi-isomorphic to its own 0-th column, which is precisely the complex $q\widehat{\Omega}_{P^\wedge/A}$, as desired. □

Definition 4.2 We let (A, I, η) be a q-divided power ring that is $(q-1)$-adically complete and separated with $I = 0$ and A topologically finitely presented and formally smooth over $\mathbf{Z}[1/N][\![q-1]\!]$-smooth. For a formally smooth topologically finitely presented A-scheme X, $q\mathrm{Cris}(X/A)$ denotes the full subcategory of $q\mathrm{Cris}(A)$ consisting of those objects (B, J, δ) in $q\mathrm{CRIS}(A)$ such that $\mathrm{Spf}(B/J)$ is a formal affine open of X as $(q-1)$-adic formal schemes over A.

We endow $q\mathrm{Cris}(X/A)$ with the Grothendieck topology induced from that of $q\mathrm{CRIS}(A)$. We call the site the *q-crystalline site* of X over A.

We note that indeed $q\mathrm{Cris}(X/A)$ is a site, the only nontrivial axiom to be checked being stability of coverings under pullbacks, which is true because q-divided power structures Zariski localize.

Define the functor

$$u_{X/A} : \mathrm{Cris}(X/A) \to \mathrm{Zar}(X)$$

to send an object (B, J, δ) to $\mathrm{Spf}(B/J) \to X$ over A. This is continuous and can be checked to have exact inverse image, thus giving a morphism between the respective topoi.

We write:

$$q\Omega_{X/A} := Ru_{X/A.*}\mathscr{O}_{q\text{-cris}}$$

where $\mathscr{O}_{q\text{-cris}}$ denotes the structure sheaf of $q\mathrm{Cris}(X/A)$ obtained by assigning to any object (B, J, δ) the value B (an assignment that is already a sheaf). This is an intrinsic notion of q-de Rham complex.

It is a mostly formal task to check that the affine result in Theorem 4.1 implies that for arbitrary X as in Definition 4.2, we have

$$q\Omega_{X/A} \otimes^{\mathbf{L}}_A A/(q-1) \simeq \Omega^*_{X_0/A_0} \tag{4.0.1}$$

where X_0 is the fiber at $q = 1$ of X (and likewise for A_0).

Proposition 4.3 *Let X_0 be a proper and smooth scheme over $\mathbf{Z}[1/N]$. Then for $X := X_0 \times_{\mathbf{Z}[1/N]} \mathbf{Z}[1/N][\![q-1]\!]$, we have that $\mathbb{H}^i(X_{\mathrm{Zar}}, q\Omega_{X/\mathbf{Z}[1/N][\![q-1]\!]})$ is a finitely generated $\mathbf{Z}[1/N][\![q-1]\!]$-module for every i.*

Proof. Let $A := \mathbf{Z}[1/N][\![q-1]\!]$. The complex $q\Omega_{X/A}$ is a derived $(q-1)$-adically complete complex of A-modules. It follows that $\mathbb{H}^i(X_{\mathrm{Zar}}, q\Omega_{X/A})$ is derived $(q-1)$-adically complete. From Theorem 4.1 and its global consequence (4.0.1), we have the quasi-isomorphism

$$q\Omega_{X/A} \otimes^{\mathbf{L}}_A (A/(q-1)A) \simeq \Omega^*_{X_0/A_0}.$$

This yields short exact sequences:

$$0 \to \mathbb{H}^i(X_{\mathrm{Zar}}, q\Omega_{X/A})/(q-1)A \to H^i_{\mathrm{dR}}(X_0) \to \mathbb{H}^{i+1}(X_{\mathrm{Zar}}, q\Omega_{X/A})[q-1] \to 0$$

for all i. The middle term is finite generated due to properness of X_0, so its submodule $\mathbb{H}^i(X_{\mathrm{Zar}}, q\Omega_{X/A})/(q-1)A$ is finitely generated. Thus, by derived Nakayama, $\mathbb{H}^i(X_{\mathrm{Zar}}, q\Omega_{X/A})$ is $(q-1)$-adically complete in the classical sense and hence is finitely generated as an A-module. □

The preceding proof indicates the sense in which the finitely generated $\mathbf{Z}[1/N][\![q-1]\!]$-module $\mathbb{H}^i(X_{\mathrm{Zar}}, q\Omega_{X/\mathbf{Z}[1/N][\![q-1]\!]})$ is a deformation of $H^i_{\mathrm{dR}}(X_0)$ up to controlled $(q-1)$-power torsion.

Remark 4.4 With further work, appropriate completions of the complex $q\Omega_{X/A}$ and its hypercohomology recover purely p-adic constructions from [**BMS**] for the p-adic completion of X_0.

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